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Periodic oscillations for a nonlinear suspension bridge model

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Abstract

We look for time-periodic solutions of the suspension bridge equation. Lazer and McKenna showed that for a certain configuration of the parameters, one may expect the existence of large-amplitude periodic solutions having the same period as the forcing term. We prove the existence of large-scale subharmonic solutions.

Keywords: Periodic solutions; Poincaré–Birkhoff; Suspension bridges

1. Introduction

Right from its opening, in July 1940, the Tacoma Narrows Bridge at Puget Sound in the state of Washington showed a clear tendency to strongly oscillate in the wind. The oscillations were of vertical type, and appeared under very different wind conditions. Even if, in some cases, they were seen to reach an amplitude of 1.5 meters, these vertical oscillations were at first considered as “benign”, since they always damped down without provoking damages to the bridge itself.

On the 7th of November of the same year, the vertical oscillations appeared from the early morning and continued until about 10 am. The stresses on the structure then forced a sudden slipping of the cable band at the center of one of the two main cables, and, at once, the type of oscillation changed, becoming of torsional type (cf. [2]). The torsional oscillations grew more and more violent, reaching angle amplitudes of 45°, until, at 11:10 am, the bridge collapsed.

The main feature of the Tacoma Bridge was its extreme flexibility, which largely depassed those of any other suspension bridge built since then. After the disaster, the engineers turned their attention towards other slender bridges which presented vertical oscillations (cf. [30]). The Bronx–Whitestone Bridge, constructed in 1939, even if not as flexible as the Tacoma Bridge,

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had showed large-amplitude vertical oscillations, and was finally strengthened in 1946. The Golden Gate Bridge, constructed in 1937, which also showed a tendency to oscillate (in 1951, during a strong gale, it had been shaken by vertical oscillations with an amplitude of 3.3 meters), was finally strengthened as well.

An easy explanation of the Tacoma Bridge disaster has been given by the resonance phenomenon (cf. [10]). The wind stream, passing through the bridge structure, produced vortices at regular intervals, alternating positive and negative pressures on the road belt. It was claimed that the frequency of the vortices was near the resonance frequency of the structure, and large-amplitude oscillations appeared like those observable for a linear harmonic oscillator.

Recently, Lazer and McKenna [39,40,41] claimed that the vertical oscillations observed in this kind of bridges could be of a nonlinear nature, due to the asymmetry of the forces involved. They proposed a nonlinear model, where the central span of a suspension bridge is considered as a one-dimensional beam suspended by linear springs, subjected to the gravitational force and to a periodic forcing due to the wind. The main feature of this model is that the springs are settled only above the beam, reacting to downward displacements but not at all to upward ones. They proved that, in some cases, one could expect large-amplitude periodic solutions.

The model we want to study is analogous to Lazer and McKenna's. Let $v(t, x)$ be the downward displacement of the bridge at the point x and time t , and denote, for any real number α , by α^+ its positive part (i.e., α^+ is equal to α when α is positive, and to 0 when α is negative). We consider the partial differential equation

$$m \frac{\partial^2 v}{\partial t^2} + d \frac{\partial v}{\partial t} + EI \frac{\partial^4 v}{\partial x^4} + \kappa [v + h]^+ = mg + F(t, x),$$

with the boundary condition

$$v(t, 0) = v(t, L) = \frac{\partial^2 v}{\partial x^2}(t, 0) = \frac{\partial^2 v}{\partial x^2}(t, L) = 0, \quad t \in \mathbb{R}.$$

Here, m is the mass per unit length, d the friction coefficient, E Young's modulus, I the moment of inertia of the cross section, κ the elastic coefficient of the cables, h the height at which the cables get loose, g the acceleration of gravity at earth's surface, F the time-periodic forcing term due to the wind and L the length of the bridge.

The bridge is supposed to be straight horizontal when no forcing is applied except the gravitational field; in other words, if $F \equiv 0$, we want the equilibrium solution to be $v \equiv 0$. Consequently,

$$\kappa = \frac{mg}{h}.$$

Unfortunately, we are not able to deal with the partial differential equation introduced above. In order to simplify the problem, following Lazer and McKenna, we introduce a new model, where it is possible to find a no-node solution of the type $v(t, x) = u(t)\sin(\pi x/L)$, with the hope that some of the properties of the solutions of the new model will reflect in those of the original one. We replace, in the above, g by $g \sin(\pi x/L)$, h by $h \sin(\pi x/L)$, and assume

the forcing to be of the form $F(t, x) = me(t)\sin(\pi x/L)$, for some periodic function $e(t)$. Setting

$$\lambda = \frac{EI\pi^4}{mL^4}, \quad \delta = \frac{d}{m},$$

one has that $u(t)$ solves the following ordinary differential equation:

$$u''(t) + \delta u'(t) + \lambda u(t) + g\left[(h^{-1}u(t) + 1)^+ - 1\right] = e(t). \quad (1.1)$$

We will look for time-periodic solutions of (1.1). Lazer and McKenna already showed that for a certain configuration of the parameters one may expect the existence of large-amplitude periodic solutions having the same period of the forcing term $e(t)$. Our attention will instead be directed in proving the existence of large-amplitude subharmonic solutions, i.e., periodic solutions having as period an integer multiple of the forcing's period.

We will prove that, if $e(t)$ is a periodic function with mean value zero, $\delta = 0$ and λ is small enough, the above equation has large-amplitude subharmonic solutions. The appearance of this type of solutions is not related to the period nor to the amplitude of the forcing $e(t)$, and in this regard they seem to well simulate the behaviour of oscillating bridges. By a numerical simulation we will show that, for a long and flexible bridge, the coefficient λ is in fact sufficiently small, and subharmonic solutions can be seen. Notice that subharmonic solutions for an equation like (1.1) had already been observed numerically in [31] by a different approach, and a theoretical explanation was asked for this phenomenon.

There is a large literature on the existence of periodic solutions to ordinary differential equations, and we can only quote a few papers or books which we think are more related to ours; cf. [4,9,12,13,15–17,19–28,31,33,34,37,39–46,48,49,51,53,55,56]. Various methods of proof have been developed. Topological degree theory may be applied when considering an equivalent fixed-point problem in a suitable space of functions, cf. [43]. Variational methods may be used for finding critical points of the associated action functional, cf. [45,52]. However, a more classical approach, going back to Poincaré, consists in searching fixed points of the map which to the points in the phase-plane makes correspond their translation along the trajectories of the differential equation (the so-called Poincaré map), cf. [37]. It is the latter method that we will exploit here, in connection with the Poincaré–Birkhoff fixed-point theorem.

The Poincaré–Birkhoff theorem, named also the “twist theorem” or the “Poincaré’s last geometric theorem”, in its original formulation, due to Poincaré [50], asserts the existence of at least two fixed points for an area-preserving homeomorphism ϕ of a planar annulus $\mathcal{A} = B[0, S] \setminus B(0, R)$ onto itself, such that the points of the inner boundary C_R are advanced along C_R in the clockwise sense and the points of the outer boundary C_S are advanced along C_S in the counterclockwise sense. This result, conjectured by Poincaré, who checked its validity in various special cases, was proved by Birkhoff [5], with respect to the existence of at least one fixed point and of two fixed points for a “generic” situation. The proof of the existence of the second fixed point in any case was obtained by Birkhoff [7], who also replaced the condition about the preservation of the areas with a more general assumption of topological nature. Due to the skepticism of some mathematicians on the validity of Birkhoff’s argument in the proof of the existence of a second fixed point, Brown and Neumann [11] were led to a very careful and

detailed checking of Birkhoff's proof, showing in a very reasonable manner its correctness. Applications of the twist theorem to dynamical systems problems coming from nonlinear mechanics and geometry were already suggested by Poincaré [50] and studied by Birkhoff [6,8].

In the case of planar nonautonomous ordinary differential equations, when one looks for the existence of periodic solutions or subharmonic solutions via the search of the fixed points of the Poincaré map or of its iterates, respectively, a major difficulty in the application of the Poincaré–Birkhoff theorem in the version stated above is the construction of annular regions which are invariant under these transformations. Hence, variants of this fixed point theorem in which the invariance conditions for the annulus and its inner and outer boundaries are not assumed became necessary for the applications. Birkhoff himself, motivated by different dynamical applications, was interested in proving some extensions of his theorem along these directions. In particular, in [6,7], he showed that his proof worked also when the annulus is not necessarily invariant under ϕ but its inner boundary is still rotated onto itself by the area-preserving homeomorphism.

Further variants of the twist theorem were proposed by other authors in more recent years. In [34], Jacobowitz, dealing with a second-order superlinear equation, used a version of the Poincaré–Birkhoff fixed-point theorem in which the inner boundary of the annulus degenerates to a fixed point of ϕ (the origin), so that now the annulus is a punctured disc and $\phi : B[0, S] \setminus \{0\} \rightarrow \mathbb{R}^2 \setminus \{0\}$ rotates in different angular directions the points near 0 and those of C_S . Moreover, C_S is not asked to be invariant under the action of ϕ . The proof of this variant of the twist theorem follows very closely Birkhoff's one [5]. It seems interesting to remark that Poincaré himself [50] already suggested a variant of his geometric theorem in which the inner circle would be shrunked to a point. Finally, in [17], Ding stated and proved a further generalization of the twist theorem that we state here for the reader's convenience (in a slightly less general form).

Let $\phi : B[0, S] \rightarrow \mathbb{R}^2$ be an area-preserving homeomorphism such that, for $\mathcal{A} = B[0, S] \setminus B(0, R)$, the following conditions hold:

$$\phi(\mathcal{A}) \subset \mathbb{R}^2 \setminus \{0\}, \quad \phi^{-1}(0) \in B(0, R).$$

On the universal covering space $\mathcal{S} = \{(\theta, \rho) : \theta \in \mathbb{R}, R \leq \rho \leq S\}$, with the standard covering projection $\Pi : (\theta, \rho) \mapsto (\rho \cos \theta, \rho \sin \theta)$, consider a lifting of ϕ of the form

$$h(\theta, \rho) = (\theta + \gamma(\theta, \rho), \eta(\theta, \rho)),$$

with $\gamma(\cdot, \rho)$ and $\eta(\cdot, \rho)$ 2π -periodic. Assume that the twist condition

$$\gamma(\theta, R)\gamma(\theta, S) < 0, \quad \forall \theta \in \mathbb{R},$$

holds. Then ϕ has a fixed point z in \mathcal{A} (indeed at least two fixed points) such that

$$\gamma(\Pi^{-1}(z)) = 0.$$

The proof of Ding's theorem reduces to Jacobowitz's one after some modification of ϕ . A more general statement can be found in [18]. See also [28] for a similar result obtained more directly from the original version of the Poincaré–Birkhoff theorem.

This paper is organized as follows. In Section 2 we present our main theoretical results, which improve some analogous theorems obtained in [16,24].

In Section 3 we develop some numerical results. Looking at fixed points of the Poincaré map, by an iterative method we detect the subharmonic solutions predicted by the theorems of Section 2. We show that these solutions are not much affected by the period or the amplitude of the forcing term $e(t)$. Our computations are done with realistic parameters for a suspension bridge like the Tacoma Narrows Bridge.

In Sections 4 and 5 we give the proof of the theorems stated in Section 2. The method of proof consists, like in the numerical illustration, in finding fixed points of the Poincaré map. To this aim, we use the generalized version of the Poincaré–Birkhoff theorem due to Ding stated above.

In Section 6 we provide some related results whose proofs differ little from those given before. We then conclude with some final remarks on the nature of the results obtained, and suggestions for further investigation.

2. Statement of the main results

In this section, we will state our main existence results, which will be proved in Sections 4 and 5. In order to understand the type of results we want to prove, it is useful to first give a look at the unforced equation

$$u''(t) + \lambda u(t) + g \left[(h^{-1}u(t) + 1)^+ - 1 \right] = 0. \tag{2.1}$$

Let us fix $A > 0$ and consider a solution $u(t)$ of (2.1) such that

$$(u(0), u'(0)) = (A, 0).$$

The solution $u(t)$ is necessarily periodic, symmetric with respect to the $u' = 0$ axis and A is its maximum value. By a phase-plane analysis, it is possible to give an explicit expression for $u(t)$. In this way, denoting by $\tau(A)$ the minimal period of $u(t)$, we have that, for $A \in (0, h]$,

$$\tau(A) = 2\pi \sqrt{\frac{h}{g + \lambda h}}$$

(the linear case), while, for $A > h$,

$$\tau(A) = 2 \left[\sqrt{\frac{h}{g + \lambda h}} \left(\frac{1}{2}\pi + \arcsin\left(\frac{h}{A}\right) \right) + \frac{1}{\sqrt{\lambda}} \arctan \sqrt{\frac{\lambda(A^2 - h^2)}{h(g + \lambda h)}} \right].$$

The other point at which $u(t)$ crosses the axis $u' = 0$ is found to be

$$(u(\frac{1}{2}\tau(A)), u'(\frac{1}{2}\tau(A))) = (-A, 0),$$

when $A \in (0, h]$, and

$$(u(\frac{1}{2}\tau(A)), u'(\frac{1}{2}\tau(A))) = \left(-\frac{g + \lambda h}{h} \left[1 + \sqrt{1 + \frac{\lambda(A^2 - h^2)}{h(g + \lambda h)}} \right], 0 \right),$$

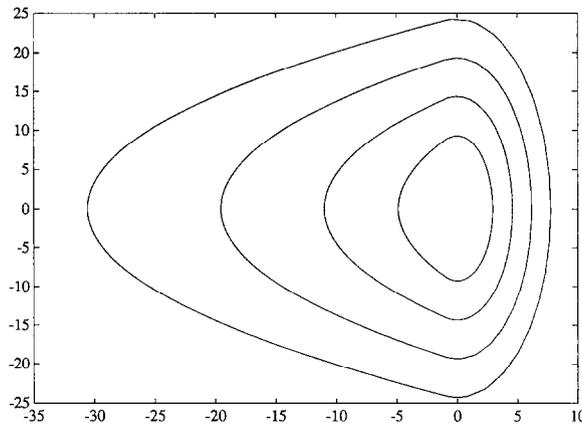


Fig. 1

when $A > h$. It is easily seen that the minimal period function $\tau(A)$ is strictly increasing for $A > h$. The limit as $A \rightarrow \infty$ of $\tau(A)$ can be finite or infinite, according to whether $\lambda > 0$ or $\lambda = 0$, respectively. In Fig. 1 we have plotted the solutions of (2.1) of periods 3, 4, 5 and 6 when $\lambda = 0$, $g = 9.8$ and $h = 1$ (the choice of the parameters will be explained in Section 3). In this case, the least minimal periods of the solutions are (slightly) greater than 2.

In what follows we will show that when a periodic forcing term is added, there is a family of subharmonic solutions which in some sense look like the solutions of the unforced equation. For example, with the above choice of the parameters and a forcing term of period 1, we may expect the existence of subharmonic solutions with periods ≥ 3 . This is exactly what will be seen numerically in Section 3. Accordingly, the theoretical results we are going to state and prove will tell us that subharmonic solutions with a sufficiently large period can be found. When $\lambda > 0$, there will be a limitation from above on the periods of the subharmonic solutions, as is to be expected from the fact that, in this case, the function $\tau(A)$ itself is bounded from above.

We will consider the more general differential equation

$$u''(t) + \lambda u(t) + g(t, u(t)) = e(t), \quad (2.2)_\lambda$$

which depends upon a nonnegative real parameter λ . Here, $e: \mathbb{R} \rightarrow \mathbb{R}$ is a locally integrable T -periodic function, $T > 0$, and $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function, T -periodic in its first variable, i.e.,

- for any $u \in \mathbb{R}$, $g(\cdot, u)$ is measurable and T -periodic;
- for almost every $t \in \mathbb{R}$, $g(t, \cdot)$ is continuous;
- for every $r > 0$ there is a $\eta_r \in L^1(0, T)$ such that

$$|g(t, u)| \leq \eta_r(t),$$

for almost every $t \in [0, T]$ and every $|u| \leq r$.

Without loss of generality, we will assume the mean value of $e(t)$ over a period to be equal to zero:

$$(i_0) \quad \frac{1}{T} \int_0^T e(t) dt = 0.$$

This can be achieved by subtracting the mean value of $e(t)$ from both sides of the equation, and redefining e and g . We will prove the following theorem.

Theorem 2.1. *Assume (i_0) and the following conditions:*

$$(i_1) \quad \lim_{u \rightarrow -\infty} \frac{g(t, u)}{u} = 0,$$

uniformly for almost every $t \in [0, T]$;

$$(i_2) \quad \int_0^T \limsup_{u \rightarrow -\infty} g(t, u) dt < 0 < \int_0^T \liminf_{u \rightarrow +\infty} g(t, u) dt.$$

Then, there is an integer $k^ \geq 1$ such that, for every $k \geq k^*$ and $\lambda \geq 0$ satisfying*

$$\lambda < \left(\frac{\pi}{kT} \right)^2, \tag{2.3}$$

$(2.2)_\lambda$ has a periodic solution $u_{\lambda,k}$ with minimal period kT having exactly two zeros in $[0, kT)$. For any sequence (u_n) of such solutions ($u_n = u_{\lambda_n, k_n}$), if $k_n \rightarrow \infty$ (and hence $\lambda_n \rightarrow 0$) as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \min_{t \in \mathbb{R}} [|u_n(t)| + |u'_n(t)|] = +\infty.$$

Remark 2.2 Condition (i_1) together with (2.3) means, roughly speaking, that the nonlinearity has to stay below the asymptote of the first curve of the Fučík spectrum (see [29]). Notice that no growth restriction is required on $g(t, u)$ when $u > 0$. In other words, the nonlinearity is even allowed to be superlinear from one side. Condition (i_2) is the well-known Landesman–Lazer condition (cf. [38]). In order for (i_2) to make sense, we have implicitly assumed that there is a L^1 -function $\gamma(t)$ for which

$$\text{sgn}(u)g(t, u) \geq \gamma(t),$$

for almost every $t \in [0, T]$ and every $u \in \mathbb{R}$.

When the function $g(t, u)$ is independent of t , the equation

$$u''(t) + \lambda u(t) + g(u(t)) = e(t) \tag{2.4}_\lambda$$

can be considered as a perturbation of an autonomous equation, and one may expect to obtain existence results under assumptions on the potential $G(u) = \int_0^u g(\xi) d\xi$, instead of the nonlinearity itself. In fact, Theorem 2.1 can in this case be generalized, and we have the following result, where the Landesman–Lazer condition is replaced by a condition first introduced in [1].

Theorem 2.3. Assume (i_0) and the following conditions:

$$(h_1) \quad \lim_{u \rightarrow -\infty} \frac{G(u)}{u^2} = 0,$$

$$(h_2) \quad \exists d > 0, \quad |u| \geq d \Rightarrow g(u)u > 0,$$

$$(h_3) \quad \lim_{|u| \rightarrow \infty} G(u) = +\infty.$$

Then, the same conclusion of Theorem 2.1 holds for equation $(2.4)_\lambda$.

The existence of T -periodic solutions for $(2.2)_\lambda$ and $(2.4)_\lambda$ under the assumptions of Theorems 2.1 and 2.3, is well known. In fact, the following proposition can be proved (cf. [27,44]).

Proposition 2.4. Under the assumptions of Theorem 2.1, for every $\lambda \in [0, (\pi/T)^2)$ Eq. $(2.2)_\lambda$ has a T -periodic solution u_λ . Moreover, for any $\lambda^* \in [0, (\pi/T)^2)$ there is a constant $K^* > 0$ such that every T -periodic solution u of $(2.2)_\lambda$ with $\lambda \in [0, \lambda^*]$ verifies

$$\sup_{t \in [0, T]} \{|u(t)| + |u'(t)|\} \leq K^*.$$

The same is true for $(2.4)_\lambda$ under the assumptions of Theorem 2.3.

3. Numerical evidence

In this section we show that the theoretical results stated in Section 2 are confirmed by a numerical approach. In this way, we are also able to evaluate the range of application of our theory. In fact, we investigate (1.1), with parameters corresponding to a realistic suspension bridge, and show that subharmonic solutions of the type predicted by Theorem 2.1 naturally appear.

We have used for our computations an Apple personal computer of the type Macintosh II fx. The method of investigation is the following. We fix an integer $k \geq 2$, and suggest to the computer a starting point in the phase-plane at time 0. By an adaptive Runge–Kutta method, the computer finds the end point of the solution at time kT . At this point, a modified Newton-type iterative procedure starts (cf. [14]), in order to find a zero for the distance between the end point and the starting point. If a zero is found, it is a fixed point of the Poincaré map, and we have only to check if it really corresponds to a periodic solution with minimal period kT , or to an iteration of a periodic solution with a smaller period.

We consider (1.1) with $e(t)$ being of sinusoidal type. The equation to be studied is the following:

$$u''(t) + \delta u'(t) + \lambda u(t) + g[(h^{-1}u(t) + 1)^+ - 1] = \alpha \sin(2\pi \nu t). \quad (3.1)$$

The evaluation of the above parameters in practical situations seems to be a difficult task. This is why we will guess a basic configuration of the parameters, and then see what happens while changing them one at a time.

Table 1

	Tacoma Narrows	Golden Gate	Bronx–Whitestone
m	$8.5 \cdot 10^3 \text{ kg m}^{-1}$	$3.1 \cdot 10^4 \text{ kg m}^{-1}$	$1.6 \cdot 10^4 \text{ kg m}^{-1}$
I	0.2 m^4	5.3 m^4	0.4 m^4
L	855 m	1280 m	700 m
λ	$7 \cdot 10^{-4} \text{ s}^{-2}$	$1 \cdot 10^{-3} \text{ s}^{-2}$	$2 \cdot 10^{-3} \text{ s}^{-2}$

All the quantities are measured in the mks-system. The acceleration of gravity at earth's surface and the steel's modulus of Young are taken to be

$$g = 9.8 \text{ m s}^{-2}, \quad E = 2 \cdot 10^{11} \text{ kg m}^{-1} \text{ s}^{-2}.$$

The value of δ is rather small for a suspension bridge, and we will at first take $\delta = 0$.

We evaluate the value of $\lambda = EI\pi^4/mL^4$ for the Tacoma Narrows Bridge and the Golden Gate and Bronx–Whitestone Bridges as they were in 1940 (cf. [2]), see Table 1. So, the value of λ for the above bridges is about 10^{-3} , and we will at first take $\lambda = 0$.

We do not have a precise idea of the value of h , but from some practical considerations we believe that it could be about 1 meter. So, we will at first take $h = 1$.

For the forcing term, we begin by considering an amplitude $\alpha = 10$ and a frequency of 1 second, i.e. $\nu = 1$.

Our basic configuration is then the following:

$$\delta = 0, \quad \lambda = 0, \quad h = 1, \quad \alpha = 10, \quad \nu = 1.$$

With this configuration of the parameters, we are able to find subharmonic solutions of periods ≥ 3 . In Fig. 2 we have visualized those of periods from 3 to 6. The solutions are symmetric with respect to the x -axis, because of the symmetry properties of the forcing term we have chosen.

We begin varying α . Plotting the subharmonic solutions of periods from 3 to 6 for $\alpha = 1$, we get a picture which almost exactly corresponds to Fig. 1, obtained for $\alpha = 0$. Figs. 3 and 4 show the subharmonic solutions of periods from 3 to 6 when $\alpha = 20$ and 50, respectively.

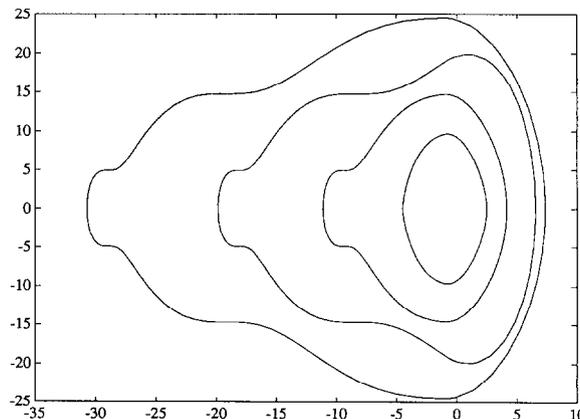


Fig. 2

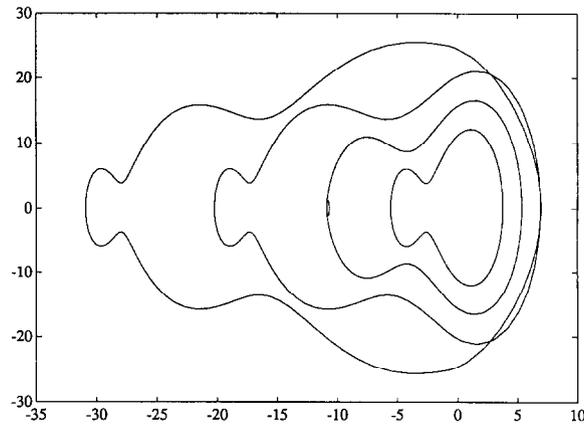


Fig. 3

We now change h . In Figs. 5 and 6 we show what happens when $h = 0.5$ and $h = 5$, respectively. One can see that for $h = 0.5$ a subharmonic solution of period 2 appears, while for $h = 5$ only subharmonics with periods ≥ 5 exist. In Fig. 5 we have plotted the subharmonics of periods from 2 to 6 when $h = 0.5$, and in Fig. 6 the subharmonics of periods 5 and 6 when $h = 5$.

We now vary λ . In Fig. 7 we see the subharmonic solutions of period 3 when $\lambda = 0, 1$ and 2. It is possible to see that if λ increases to about 2.25 (where approximately we touch the first Fučík curve), the amplitudes tend to infinity.

We now modify δ . The subharmonics seem to survive when δ is taken positive and small. In Fig. 8 we consider the subharmonics of period 3, when $\delta = 0, 0.03, 0.06$. Notice that the solutions are no more symmetric in this case. While increasing δ , we did not succeed in making our iteration converge any more.

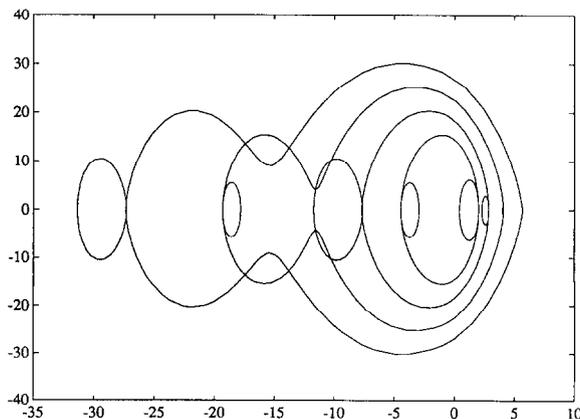


Fig. 4

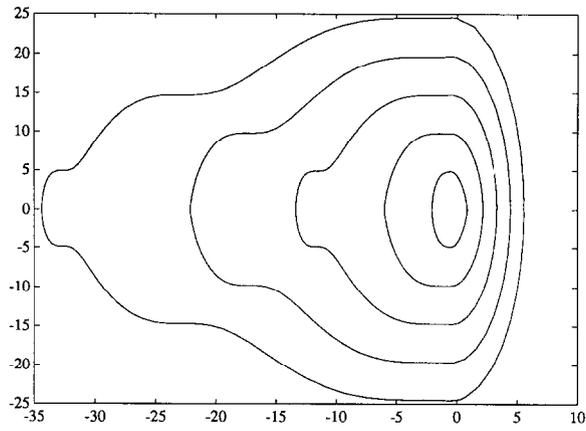


Fig. 5

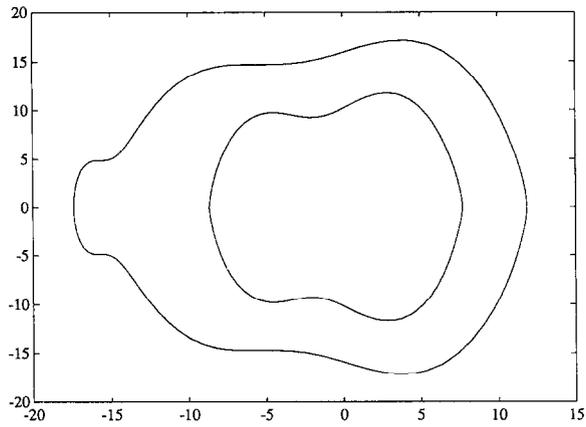


Fig. 6

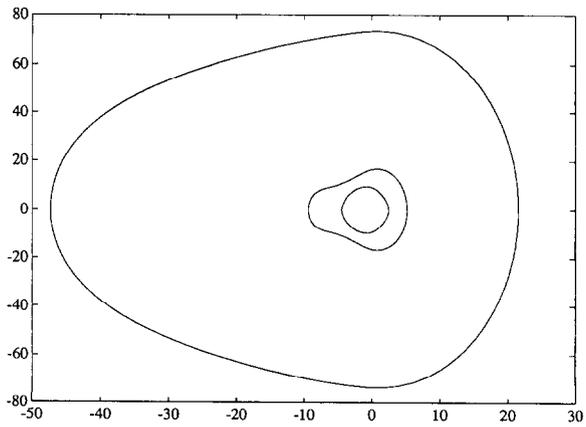


Fig. 7

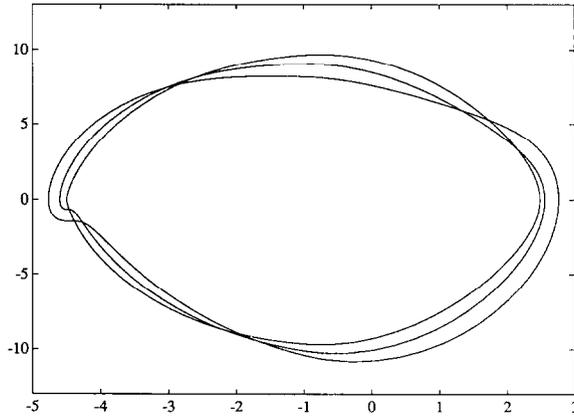


Fig. 8

We now change ν . Let, for instance $\nu = 3$. In this case, we only see subharmonic solutions of orders ≥ 7 , i.e., of periods $\geq \frac{7}{3}$. In Fig. 9 we show the subharmonic solutions of order 7, 8 and 9 when $\nu = 3$. Notice that the amplitude of the subharmonic solutions of order 9 is about the same as the one of order 3 when $\nu = 1$. An analogous statement can be seen to be true for the subharmonic solution of order 6 when $\nu = 2$.

In conclusion, the subharmonic solutions computed numerically all have exactly two zeros, and the minimal periods as well as the amplitudes are seen to behave as predicted by the theorems stated in Section 2.

4. Proof of Theorem 2.1

In order to prove Theorem 2.1, we will look for fixed points of the Poincaré map, which assigns to each point in the phase-plane its translation along a solution of the differential

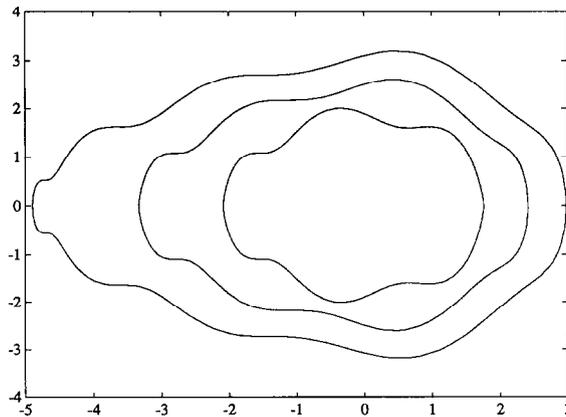


Fig. 9

equation (see [37]). Notice that the same procedure has been followed numerically in Section 3. Here, instead of an iterative method, we will make use of the generalized version of the Poincaré–Birkhoff theorem due to Ding, stated in Section 1.

We will first write $(2.2)_\lambda$ as an equivalent equation (see $(4.1)_\lambda$ below), which is more suitable for our analysis. Then, for each point of the phase-plane $z_0 \in \mathbb{R}^2$, we denote by $x(t; z_0)$ the solution which is such that

$$(x(0; z_0), x'(0; z_0)) = z_0.$$

The Poincaré map ϕ_k relative to the time kT is then defined as

$$\phi_k(z_0) = (x(kT; z_0), x'(kT; z_0)).$$

Notice that ϕ_k is the k th iterate of ϕ_1 . In order for the Poincaré map to be well-defined, we want the solutions to Cauchy problems to be unique and globally defined.

Concerning uniqueness, we observe that it is possible to use a standard approximation of the nonlinearity by smooth functions, and to work on the approximated equations, for which uniqueness holds. This procedure has been explained in [37] and developed with full details in [16] for equations like the one we are considering. Hence, for simplicity, we prefer avoiding this technical point, and, without loss of generality, from now on we assume the uniqueness for the Cauchy problems associated to $(2.2)_\lambda$.

Under the uniqueness condition, the Poincaré map is continuous on its domain of definition. If it is globally defined, the Poincaré map is a homeomorphism on \mathbb{R}^2 .

The global existence will be proved in Lemma 4.5 below. We will show that, because of (i_0) and (i_2) , the solutions of the Cauchy problems cannot blow up in finite time without performing an infinite number of rotations in the phase-plane (see [16,21,33] for a similar approach). On the other hand, (i_1) and (2.3) give a uniform lower estimate of the time needed for a solution to perform a rotation while being far away from the origin. In other words, the time-map has a positive inferior limit at infinity. These two considerations are in contradiction with a possible blow-up in finite time of the solutions.

As a direct consequence of Liouville’s theorem, we have that the Poincaré map is area-preserving (cf. [54]). In order to apply the generalized Poincaré–Birkhoff theorem stated in Section 1, as a final step we will have to prove that the twist condition is satisfied on a suitable annulus.

We start by making a change of variable, which transforms $(2.2)_\lambda$ in an equivalent one for which 0 is a trivial solution. Moreover, we fix a certain $\lambda^* \in [0, (\pi/T)^2)$, so that we can make use of the uniform estimates of Proposition 2.4.

Let $x = u - u_\lambda$, where u_λ is a T -periodic solution of $(2.2)_\lambda$ given by Proposition 2.4, and set

$$f(t, x, \lambda) = \lambda(x + u_\lambda(t)) + g(t, x + u_\lambda(t)), \quad h(t, \lambda) = e(t) - u_\lambda''(t).$$

Then, $(2.2)_\lambda$ is equivalent to

$$x''(t) + f(t, x(t), \lambda) = h(t, \lambda). \tag{4.1}_\lambda$$

Here, the functions $f : \mathbb{R} \times \mathbb{R} \times [0, (\pi/T)^2) \rightarrow \mathbb{R}$ and $h : \mathbb{R} \times [0, (\pi/T)^2) \rightarrow \mathbb{R}$ satisfy the following conditions:

- for any $x \in \mathbb{R}$ and $\lambda \in [0, (\pi/T)^2)$, the functions $f(\cdot, x, \lambda)$ and $h(\cdot, \lambda)$ are measurable and T -periodic;

- for almost every $t \in \mathbb{R}$, $f(t, \cdot, \cdot)$ and $h(t, \cdot)$ are continuous;
- for every $r > 0$ there is a $\nu_r \in L^1(0, T)$ such that

$$|h(t, \lambda) - f(t, x, \lambda)| \leq \nu_r(t),$$

for almost every $t \in [0, T]$, every $|x| \leq r$ and every $\lambda \in [0, \lambda^*]$.

Moreover, we have

$$(j_0) \quad \frac{1}{T} \int_0^T h(t, \lambda) dt = 0,$$

for every $\lambda \in [0, (\pi/T)^2]$;

$$(j_1) \quad \lim_{x \rightarrow -\infty} \frac{f(t, x, \lambda)}{x} = \lambda,$$

uniformly with respect to $t \in \mathbb{R}$ and $\lambda \in [0, \lambda^*]$,

$$(j_2) \quad \sup_{\lambda \in [0, \lambda^*]} \int_0^T \limsup_{x \rightarrow -\infty} f(t, x, \lambda) dt < 0 < \inf_{\lambda \in [0, \lambda^*]} \int_0^T \liminf_{x \rightarrow +\infty} f(t, x, \lambda) dt,$$

$$(j_3) \quad f(t, 0, \lambda) = h(t, \lambda),$$

for almost every $t \in \mathbb{R}$ and every $\lambda \in [0, (\pi/T)^2]$.

Since $(4.1)_\lambda$ has the trivial solution 0, by the uniqueness assumption for the Cauchy problems, a nonzero solution will at no point hit the origin. We may then introduce polar coordinates in the phase-plane, being sure that for any nontrivial solution, they will be well-determined at any time. If $x(t)$ is a nontrivial solution of $(4.1)_\lambda$, we denote by $(\rho(t), \theta(t))$ the polar coordinates of $(x(t), x'(t))$.

Lemma 4.1. For any nontrivial solution of $(4.1)_\lambda$ and any $t_0 < t_1$ in its domain, $\theta(t_1) - \theta(t_0) < \pi$.

Proof. Let $x(t)$ be a solution of $(4.1)_\lambda$. It is easy to see that, if, for some t in the domain of x , $\theta(t) = \frac{1}{2}\pi + m\pi$, $m \in \mathbb{Z}$, then $\theta'(t) < 0$. Standard results of flow invariance then imply that the sets $\{(\rho, \theta) \in \mathbb{R}^2 \mid \rho > 0, \theta \in \frac{1}{2}\pi + m\pi\}$, $m \in \mathbb{Z}$, are positively invariant, i.e., if t_0 in the domain of x is such that $\theta(t_0) \leq \frac{1}{2}\pi + m_0\pi$ for some $m_0 \in \mathbb{Z}$, then $\theta(t) \leq \frac{1}{2}\pi + m_0\pi$, for all $t \geq t_0$. The result readily follows from the above considerations. \square

Assume (j_2) . Then, there is a $d_1 \geq R$ and a Caratheodory function $l(t, \lambda)$ with the following properties:

- for every $|x| \geq d_1$ and almost every $t \in [0, T]$,

$$\text{sgn}(x)f(t, x, \lambda) \geq l(t, \lambda);$$

- there is a $\delta > 0$ such that, for every $\lambda \in [0, \lambda^*]$,

$$\frac{1}{T} \int_0^T l(t, \lambda) dt \geq \delta;$$

- there is a $M > 0$ such that, for every $\lambda \in [0, \lambda^*]$,

$$\int_0^T [|l(t, \lambda)| + h(t, \lambda)] dt \leq M.$$

The following lemma is crucial for the proof of Theorem 2.1.

Lemma 4.2. *Assume (j_0) , (j_2) and (j_3) . Then, for every $R > 0$ there is a $R_1 > R$ such that, for any $\lambda \in [0, \lambda^*]$, any solution $x(t)$ of $(4.1)_\lambda$ and any t_0 in its domain, if $\rho(t_0) \geq R_1$ and $-\frac{1}{2}\pi + m\pi \leq \theta(t_0) \leq \frac{1}{2}\pi + m\pi$, $m \in \mathbb{Z}$, there is a $t_1 > t_0$ in the domain of $x(t)$ for which $\theta(t_1) = -\frac{1}{2}\pi + m\pi$ and $\rho(t) \geq R$ for every $t \in [t_0, t_1]$.*

Proof. By the symmetry of the assumptions, it is sufficient to consider the case $x(t_0) \geq 0$. We will analyse different starting point regions in the phase-plane. Accordingly, we consider the following five situations.

Case 1. Choose $d_2 > 0$ such that

$$d_2 > d_1 \max\{1, T^{-1}\} + \|v_{d_1}\|_{L^1},$$

and assume that

$$(x(t_0), y(t_0)) \in (0, d_1] \times (-\infty, -d_2].$$

Let $t_0^{(1)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(1)})$,

$$(x(t), y(t)) \in (0, +\infty) \times (-\infty, 0).$$

We will show that $t_0^{(1)}$ is in the domain of x , $x(t_0^{(1)}) = 0$ and

$$(x(t), y(t)) \in (0, d_1] \times (-\infty, -d_1),$$

for every $t \in [t_0, t_0^{(1)})$. In this case, set $t_1 = t_0^{(1)}$.

For every $t \in [t_0, t_0^{(1)})$, since $x'(t) = y(t) < 0$, we have

$$|y(t) - y(t_0)| \leq \int_{t_0}^t v_{d_1}(s) ds,$$

and so

$$\begin{aligned} d_1 &\geq x(t_0) - x(t) = \int_t^{t_0} y(s) ds \\ &\geq -y(t_0)(t - t_0) - \int_{t_0}^t \left(\int_{t_0}^s v_{d_1}(\xi) d\xi \right) ds \\ &\geq d_2(t - t_0) + \int_{t_0}^t \left(\int_{t_0}^s v_{d_1}(\xi) d\xi \right) ds \\ &> \frac{d_1}{T}(t - t_0) + \int_{t_0}^t \left(\|v_{d_1}\|_{L^1} - \int_{t_0}^s v_{d_1}(\xi) d\xi \right) ds, \end{aligned}$$

for every $t \in [t_0, t_0^{(1)})$. It follows from the above that, for every $t \in [t_0, t_0^{(1)})$, it has to be $t \neq t_0 + T$. In other words, $t_0^{(1)} < t_0 + T$ and, more precisely,

$$t_0^{(1)} - t_0 \leq \frac{d_1}{-y(t_0) - \|\nu_{d_1}\|_{L^1}}. \tag{4.2}$$

Then, for every $t \in [t_0, t_0^{(1)})$, one has

$$|y(t) - y(t_0)| \leq \|\nu_{d_1}\|_{L^1},$$

and, on the other hand,

$$0 < x(t) \leq d_1.$$

Hence, there is no blow-up in $[t_0, t_0^{(1)})$, and

$$y(t) \leq -d_2 + \|\nu_{d_1}\|_{L^1} < -d_1,$$

for every $t \in [t_0, t_0^{(1)})$, so that $x(t_0^{(1)}) = 0$.

Case 2. Assume that

$$(x(t_0), y(t_0)) \in (d_1, +\infty) \times (-\infty, -(d_2 + M)].$$

Let $t_0^{(2)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(2)})$,

$$(x(t), y(t)) \in (d_1, +\infty) \times (-\infty, +\infty).$$

We will show that $t_0^{(2)}$ is in the domain of x , $x(t_0^{(2)}) = d_1$ and

$$(x(t), y(t)) \in (d_1, +\infty) \times (-\infty, -d_2),$$

for every $t \in [t_0, t_0^{(2)})$. Going back to case 1, we determine t_1 .

Let $n_0 \geq 1$ be the least integer such that $[t_0, t_0^{(2)}) \subset [t_0, t_0 + n_0T)$. For every $t \in [t_0, t_0^{(2)})$, we have

$$\begin{aligned} y(t) - y(t_0) &= \int_{t_0}^t [h(s, \lambda) - f(s, x(s), \lambda)] \, ds \\ &\leq \int_{t_0}^t [h(s, \lambda) - l(s, \lambda)] \, ds \\ &\leq \int_{t_0}^{t_0 + n_0T} [h(s, \lambda) - l(s, \lambda)] \, ds + M \\ &\leq -\delta n_0T + M \\ &\leq -\delta(t - t_0) + M. \end{aligned}$$

Hence, for every $t \in [t_0, t_0^{(2)})$, $y(t) < -d_2$, and

$$d_1 \leq x(t) = x(t_0) + \int_{t_0}^t y(s) \, ds \leq x(t_0) - d_2(t - t_0),$$

so that

$$t_0^{(2)} - t_0 \leq \frac{x(t_0) - d_1}{d_2}. \tag{4.3}$$

Since

$$|y(t) - y(t_0)| \leq \int_{t_0}^t \nu_{x(t_0)}(s) \, ds,$$

there is no blow-up in $[t_0, t_0^{(2)})$ and $x(t_0^{(2)}) = d_1$.

Case 3. Let $d_3 > 0$ be such that

$$d_3 > d_1 + \frac{(d_2 + M)(d_1 + d_2 + 2M)}{\delta},$$

and assume that

$$(x(t_0), y(t_0)) \in [d_3, +\infty) \times (-(d_2 + M), d_1].$$

Let $t_0^{(3)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(3)})$,

$$(x(t), y(t)) \in (d_1, +\infty) \times (-(d_2 + M), +\infty).$$

We will show that $t_0^{(3)}$ is in the domain of x , $y(t_0^{(3)}) = -(d_2 + M)$ and

$$(x(t), y(t)) \in (d_1, +\infty) \times (-(d_2 + M), d_1 + M],$$

for every $t \in [t_0, t_0^{(3)})$. Going back to case 2, we determine t_1 .

As in case 2, for every $t \in [t_0, t_0^{(3)})$, we have

$$-(d_2 + M) \leq y(t) \leq y(t_0) - \delta(t - t_0) + M,$$

so that

$$t_0^{(3)} - t_0 \leq \frac{d_1 + d_2 + 2M}{\delta}. \tag{4.4}$$

Then, we have

$$|x(t) - x(t_0)| \leq (d_2 + M)(t - t_0) \leq \frac{(d_2 + M)(d_1 + d_2 + 2M)}{\delta},$$

for every $t \in [t_0, t_0^{(3)})$. Hence, there is no blow-up in $[t_0, t_0^{(3)})$,

$$x(t) \geq d_3 - \frac{(d_2 + M)(d_1 + d_2 + 2M)}{\delta} > d_1,$$

and it follows from the above that $y(t_0^{(3)}) = -d_2 - M$.

Case 4. Assume that

$$(x(t_0), y(t_0)) \in [d_3, +\infty) \times (d_1, +\infty).$$

Let $t_0^{(4)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(4)})$,

$$(x(t), y(t)) \in (d_1, +\infty) \times (d_1, +\infty).$$

We will show that $t_0^{(4)}$ is in the domain of x , $y(t_0^{(4)}) = d_1$ and

$$(x(t), y(t)) \in [d_3, +\infty) \times (d_1, +\infty),$$

for every $t \in [t_0, t_0^{(4)})$. Going back to case 3, we determine t_1 .

For every $t \in [t_0, t_0^{(4)})$, we have

$$d_1 < y(t) \leq y(t_0) - \delta(t - t_0) + M,$$

so that

$$t_0^{(4)} - t_0 \leq \frac{y(t_0) - d_1 + M}{\delta}. \quad (4.5)$$

Moreover,

$$d_1 \leq x(t) \leq x(t_0) + \int_{t_0}^t y(s) \, ds \leq x(t_0) + (y(t_0) + M) \frac{(y(t_0) - d_1 + M)}{\delta}.$$

So, there is no blow-up in $[t_0, t_0^{(4)})$, and we have $y(t_0^{(4)}) = d_1$.

Case 5. Set

$$d_4 := \max\left\{d_1, \frac{d_3}{T}\right\} + \|\nu_{d_3}\|_{L^1},$$

and assume that

$$(x(t_0), y(t_0)) \in [0, d_3) \times (d_4, +\infty).$$

Let $t_0^{(5)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(5)})$,

$$(x(t), y(t)) \in (-\infty, d_3) \times (-\infty, +\infty).$$

We will show that $t_0^{(5)}$ is in the domain of x , $x(t_0^{(5)}) = d_3$ and

$$(x(t), y(t)) \in [0, d_3) \times (d_1, +\infty),$$

for every $t \in [t_0, t_0^{(5)})$. Going back to case 4, we determine t_1 .

This case is analogous to case 1. For every $t \in [t_0, t_0^{(5)})$, since $x'(t) = y(t) > 0$, we have

$$|y(t) - y(t_0)| \leq \int_{t_0}^t \nu_{d_3}(s) \, ds,$$

and so

$$\begin{aligned} d_3 \geq x(t) - x(t_0) &\geq y(t_0)(t - t_0) - \int_{t_0}^t \int_{t_0}^s \nu_{d_3}(\xi) \, d\xi \, ds \\ &\geq \frac{d_3}{T}(t - t_0) + \int_{t_0}^t \left(\|\nu_{d_3}\|_{L^1} - \int_{t_0}^s \nu_{d_3}(\xi) \, d\xi \right) ds, \end{aligned}$$

so that, for every $t \in [t_0, t_0^{(5)})$, it has to be $t \neq t_0 + T$, and, more precisely,

$$t_0^{(5)} - t_0 \leq \frac{d_3}{y(t_0) - \|\nu_{d_3}\|_{L^1}}. \quad (4.6)$$

So, for every $t \in [t_0, t_0^{(5)})$, $x(t) > 0$,

$$|y(t) - y(t_0)| \leq \|v_{d_3}\|_{L^1},$$

there is no blow-up in $[t_0, t_0^{(5)})$, and $x(t_0^{(5)}) = d_3$.

The proof of the lemma is now easily completed, by taking

$$R_1 \geq \sqrt{d_3^2 + \max\{(d_2 + M)^2, d_4^2\}}. \quad \square$$

Lemma 4.3. Assume (j_0) , (j_2) and (j_3) , and fix $R > 0$. Then there are R_1, R_2 and R_3 such that $R \leq R_1 < R_2 < R_3$ and, for any $\lambda \in [0, \lambda^*]$ and any solution $x(t)$ of $(4.1)_\lambda$ such that, for some t_0 in its domain, $\rho(t_0) = R_2$, one has the following alternative.

(a) For some $t_1 > t_0$ in the domain of x , $\rho(t_1) \notin (R_1, R_3)$. In this case, for every $t \geq t_1$ in the domain of x , $\theta(t) - \theta(t_0) < -2\pi$.

(b) For every $t \geq t_0$, one has $\rho(t) \in (R_1, R_3)$. Then, there is an integer $k^* \geq 1$ such that, for every $t \geq t_0 + k^*T$, $\theta(t) - \theta(t_0) < -2\pi$.

Proof. Let $R_1 \geq R$ be like in Lemma 4.2. By an iterative use of Lemma 4.2, we are able to find an $R_2 > R_1$ such that, if, for $\lambda \in [0, \lambda^*]$, a solution $x(t)$ of $(4.1)_\lambda$ is such that, for some $t_0 < t_1$ in its domain, $\rho(t_0) = R_2$ and $\rho(t_1) \leq R_1$, then $\theta(t_1) - \theta(t_0) \leq -3\pi$. Analogously, starting from R_2 and using Lemma 4.2 repeatedly for the negative flow, we find an $R_3 > R_2$ such that, if $\rho(t_0) = R_2$ and $\rho(t_1) \geq R_3$, then $\theta(t_1) - \theta(t_0) \leq -3\pi$. By Lemma 4.1, the first part of the lemma is proved.

For the second part, assume that, for every $t \geq t_0$, one has $\rho(t) \in (R_1, R_3)$. Then the time estimates (4.2)–(4.6) and their symmetries for the negative values of $x(t)$ give us a uniform lower estimate for the time needed for a solution to rotate around the origin of the phase-plane. Hence, there is an integer $k^* \geq 1$ such that $\theta(t_0 + k^*T) - \theta(t_0) \leq -3\pi$. The second part of the lemma then follows using Lemma 4.1, again. \square

As a corollary of Lemma 4.3, we have that, if a solution of $(4.1)_\lambda$ blows up in a finite time, it has to perform an infinite number of rotations.

Lemma 4.4. For every $R > 0$, for every $\lambda \in [0, \lambda^*]$ and $k \geq 1$ satisfying (2.3), there is an $S_1 > R$ with the following property: for any solution $x(t)$ of $(4.1)_\lambda$ and any $t_0 < t_1$ in its domain such that $t_1 - t_0 \leq kT$, if $\rho(t) \geq S_1$ for every $t \in [t_0, t_1]$, then $\theta(t_1) - \theta(t_0) > -2\pi$.

Proof. Assume the contrary. Let $t_2 < t_3$ in the domain of x such that, for some $m \in \mathbb{Z}$, $\theta(t_2) = -\frac{1}{2}\pi + 2m\pi$, $\theta(t_3) = -\frac{3}{2}\pi + 2m\pi$, and $-\frac{3}{2}\pi + 2m\pi < \theta(t) < -\frac{1}{2}\pi + 2m\pi$, for every $t \in (t_2, t_3)$. For any $\mu > 0$, it is possible to see that

$$\frac{1}{2} = \frac{\mu}{2\pi} \int_{t_2}^{t_3} \frac{x(t)(f(t, x(t), \lambda) - h(t, \lambda)) + (x'(t))^2}{\mu^2(x(t))^2 + (x'(t))^2} dt$$

(cf. [20]). Fix $\epsilon > 0$ such that $\lambda + \epsilon < (\pi/kT)^2$. There is a $K_\epsilon > 0$ such that, for every $x \leq 0$,

$$xf(t, x, \lambda) \leq (\lambda + \epsilon)x^2 + K_\epsilon.$$

Fix $\mu = \sqrt{\lambda + \epsilon}$. Then,

$$\begin{aligned} \frac{\pi}{\sqrt{\lambda + \epsilon}} &\leq \int_{t_2}^{t_3} \frac{(\lambda + \epsilon)(x(t))^2 + K_\epsilon + (x'(t))^2 - x(t)h(t, \lambda)}{(\lambda + \epsilon)(x(t))^2 + (x'(t))^2} dt \\ &= (t_3 - t_2) + \int_{t_2}^{t_3} \frac{K_\epsilon - x(t)h(t, \lambda)}{(\lambda + \epsilon)(x(t))^2 + (x'(t))^2} dt. \end{aligned}$$

Taking $[(\lambda + \epsilon)(x(t))^2 + (x'(t))^2]$ large enough, we get

$$\frac{\pi}{\sqrt{\lambda + \epsilon}} < kT.$$

Since the above holds for any $\epsilon > 0$, we have $(\pi/\sqrt{\lambda}) \leq kT$, in contradiction with the hypothesis. \square

Lemma 4.5. Assume (j_0) – (j_3) . Then the solutions of the Cauchy problems associated to $(4.1)_\lambda$ are globally defined in \mathbb{R} .

Proof. If a solution were not globally defined, it would have to perform an infinite number of rotations in finite time, while going to infinity. But Lemma 4.4 tells us that this is impossible, since rotating once requires at least a length of time kT . \square

Proof of Theorem 2.1. Let K^* be like in Proposition 2.4, and fix $R > \sqrt{2}K^*$ arbitrarily. Consider R_1 , R_2 and k^* given by Lemma 4.3. Fix $k \geq k^*$ and $\lambda \in [0, \lambda^*]$ satisfying (2.3). Let S_1 be like in Lemma 4.4. By the global existence for Cauchy problems, it is possible to find a $S_2 > S_1$ such that, if $x(t)$ is a solution of $(4.1)_\lambda$ for which $\rho(0) = S_2$, then $\rho(t) \geq S_1$ for every $t \in [0, kT]$.

For any $z_0(x_0, y_0)$ in the phase-plane, let $x(t; z_0)$ be the solution of $(4.1)_\lambda$ such that

$$(x(0; z_0), x'(0; z_0)) = z_0.$$

Let (θ_0, ρ_0) and $(\theta(t; z_0), \rho(t; z_0))$ be the polar coordinates of z_0 and $(x(t; z_0), x'(t; z_0))$, respectively. Setting

$$\gamma(\theta_0, \rho_0) = \theta(kT; z_0) - \theta_0 + 2\pi, \quad \eta(\theta_0, \rho_0) = \rho(kT; z_0),$$

we have that

$$h(\theta_0, \rho_0) = (\theta_0 + \gamma(\theta_0, \rho_0), \eta(\theta_0, \rho_0))$$

is a lifting of the Poincaré map relative to the time kT , and, from Lemmas 4.3 and 4.4, that the twist condition is satisfied for the annulus $B[0, S_2] \setminus B(0, R_2)$. Applying the modified version of the Poincaré–Birkhoff theorem stated in Section 1, a fixed point of the Poincaré map relative to the time kT can be found in this annulus, and hence there is a kT -periodic solution $x_{\lambda, k}$ of $(4.1)_\lambda$, which performs exactly one rotation in the phase plane, and whose trajectory starts from the annulus $B[0, S_2] \setminus B(0, R_2)$ at time $t = 0$. We may also affirm, by Lemma 4.3, that $(x_{\lambda, k}(t), x'_{\lambda, k}(t)) \notin B(0, R_1)$, for every $t \in \mathbb{R}$.

Corresponding to $x_{\lambda,k}$, there is a solution $u_{\lambda,k}$ of (2.2) $_{\lambda}$, given by $u_{\lambda,k} = x_{\lambda,k} + u_{\lambda}$. Let us prove that $u_{\lambda,k}$ has exactly two zeros in $[0, kT]$. It is possible to find $t_1 < t_2 < t_3 < t_4 < t_1 + kT$ such that

$$\begin{aligned} x_{\lambda,k}(t) &> K^*, \quad \forall t \in [t_1, t_2], \\ x'_{\lambda,k}(t) &< -K^*, \quad \forall t \in [t_2, t_3], \\ x_{\lambda,k}(t) &< -K^*, \quad \forall t \in [t_3, t_4], \\ x'_{\lambda,k}(t) &> K^*, \quad \forall t \in [t_4, t_1 + kT]. \end{aligned}$$

Because of the estimate in Proposition 2.1, one then has

$$\begin{aligned} u_{\lambda,k}(t) &> 0, \quad \forall t \in [t_1, t_2], \\ u'_{\lambda,k}(t) &< 0, \quad \forall t \in [t_2, t_3], \\ u_{\lambda,k}(t) &< 0, \quad \forall t \in [t_3, t_4], \\ u'_{\lambda,k}(t) &> 0, \quad \forall t \in [t_4, t_1 + kT]. \end{aligned}$$

Accordingly, $u_{\lambda,k}$ has exactly two zeros in $[t_1, t_1 + kT]$, hence also in $[0, kT]$.

Concerning the last part of the statement, assume, by contradiction, that there is a sequence (u_{λ_n, k_n}) of solutions of the type found above and a sequence (t_n) , $t_n \in [0, k_n T]$, such that, for a certain constant $\tilde{R}_2 > R_2$, one has

$$|u_{\lambda_n, k_n}(t_n)| + |u'_{\lambda_n, k_n}(t_n)| \leq \tilde{R}_2,$$

for every n . By Lemma 4.3, since our solutions rotate only once, there is a $\tilde{R}_3 > \tilde{R}_2$ such that, for every $t \in \mathbb{R}$,

$$\left((u_{\lambda_n, k_n}(t))^2 + (u'_{\lambda_n, k_n}(t))^2 \right)^{1/2} \leq \tilde{R}_3.$$

Therefore, the solutions lie in a fixed annulus with radii R_1 and \tilde{R}_3 . Using the second part of Lemma 4.3, we see that this is impossible, as the periods $k_n T$ tend to infinity. Theorem 2.1 is therefore completely proved. \square

5. Proof of Theorem 2.3

The proof of Theorem 2.3 follows the lines of that of Theorem 2.1. This is why we will only point out the main modifications which are needed.

The proof starts in the same way as that of Theorem 2.1. One makes a change of variable, reducing (2.4) $_{\lambda}$ to (4.1) $_{\lambda}$. In order to prove the analogues of Lemmas 4.1–4.5, we define

$$E(t) = \int_0^T e(s) \, ds,$$

and consider the following system associated to (2.4) $_{\lambda}$:

$$u' = v + E(t), \quad v' = -\lambda u - g(u). \tag{5.1}_{\lambda}$$

Lemma 4.1 does not need any change at all.

Concerning Lemma 4.2, we would like to have the same conclusion assuming conditions (h_2) and (h_3) instead of (j_2) and (j_3) . Going for the proof, we will make the needed estimates directly on system $(5.1)_\lambda$. Let K^* be like in Proposition 2.4. While working on system $(5.1)_\lambda$, the estimates on $(u(t), v(t))$ may differ from those on $(x(t), y(t))$ by at most K^* . Keeping this in mind, we take d'_1 satisfying

$$d'_1 > K^* + \max\{d, \|E\|_\infty, R\}.$$

For any $r > 0$, we define

$$v'_r = \lambda^* r + \max_{|u| \leq r} |g(u)|.$$

As previously, we examine five different cases.

Case 1'. Choose

$$d'_2 > d'_1 \max\{1, 2T^{-1}\} + v'_{d'_1} T + \|E\|_\infty,$$

and assume that

$$(u(t_0), v(t_0)) \in (-d'_1, d'_1] \times (-\infty, -d'_2].$$

Proceeding like in Lemma 4.2, one can see that there is a $t_0^{(1)} > t_0$ in the domain of u such that $u(t_0^{(1)}) = -d'_1$,

$$t_0^{(1)} - t_0 \leq \frac{2d'_1}{-v(t_0) - v'_{d'_1} T - \|E\|_\infty},$$

and, for every $t \in [t_0, t_0^{(1)})$,

$$(u(t), v(t)) \in (-d'_1, d'_1] \times (-\infty, -d'_2).$$

Case 2'. Assume

$$(u(t_0), v(t_0)) \in (d'_1, +\infty) \times (-\infty, -d'_2].$$

It is easy to see in this case that there is a $t_0^{(2)} > t_0$ in the domain of u such that $u(t_0^{(2)}) = d'_1$,

$$t_0^{(2)} - t_0 \leq \frac{u(t_0) - d'_1}{d'_2 - \|E\|_\infty},$$

and, for every $t \in [t_0, t_0^{(2)})$,

$$(u(t), v(t)) \in (d'_1, +\infty) \times (-\infty, d'_2].$$

Case 3'. Assume

$$(u(t_0), v(t_0)) \in [d'_3, +\infty) \times (-d'_2, d'_1],$$

where $d'_3 > d'_1$ is sufficiently large, to be determined. This case has to be treated with some care. Let $t_0^{(3)} > t_0$ be maximal for the property that, for every $t \in [t_0, t_0^{(3)})$,

$$(u(t), v(t)) \in (d'_1, +\infty) \times (-d'_2, +\infty).$$

We will show that $t_0^{(3)}$ is in the domain of u , $v(t_0^{(3)}) = -d'_2$ and, for every $t \in [t_0, t_0^{(3)})$,

$$(u(t), v(t)) \in (d'_1, +\infty) \times (-d'_2, d'_1].$$

For every $t \in [t_0, t_0^{(3)})$, we have

$$v(t) - \|E\|_\infty \leq u'(t) \leq v(t) + \|E\|_\infty, \quad -v'(t) = \lambda u(t) + g(u(t)) > 0.$$

Multiplying, we get

$$-vv' + \|E\|_\infty v' \leq \lambda uu' + g(u)u' \leq -vv' - \|E\|_\infty v'.$$

Integration gives

$$\begin{aligned} -\frac{1}{2}(d'_2)^2 - 2d'_2 \|E\|_\infty &\leq \frac{1}{2}((v(t_0))^2 - (v(t))^2) - \|E\|_\infty(v(t_0) - v(t)) \\ &\leq \frac{1}{2}\lambda((u(t))^2 - (u(t_0))^2) + G(u(t)) - G(u(t_0)) \\ &\leq \frac{1}{2}((v(t_0))^2 - (v(t))^2) + \|E\|_\infty(v(t_0) - v(t)) \\ &\leq \frac{1}{2}(d'_2)^2 + 2d'_2 \|E\|_\infty. \end{aligned}$$

From the above and (h_3) , one can easily see that there is no blow-up of the solutions, and if d'_3 is chosen sufficiently large, then $u(t) > d'_1$. More precisely, there is a $L_0 > d'_1$, depending on $u(t_0)$, such that

$$d'_1 \leq u(t) \leq L_0,$$

for every $t \in [t_0, t_0^{(3)})$. Then,

$$-v'(t) = \lambda u(t) + g(u(t)) \geq \min_{[d'_1, L_0]} g := \delta > 0,$$

and so,

$$t_0^{(3)} - t_0 < \frac{d'_1 + d'_2}{\delta},$$

while $v(t_0^{(3)}) = -d'_2$.

Case 4'. Assume

$$(u(t_0), v(t_0)) \in [d'_3, +\infty) \times (d'_1, +\infty).$$

Then, there is a $t_0^{(4)} > t_0$ in the domain of u such that $v(t_0^{(4)}) = d'_1$ and, for every $t \in [t_0, t_0^{(4)})$,

$$(u(t), v(t)) \in [d'_3, +\infty) \times (d'_1, +\infty).$$

In fact, since $u'(t) \geq 0$, working as in case 4, we are able to find a constant $L'_0 > d'_3$ such that

$$d'_3 \leq u(t) \leq L'_0,$$

for every $t \geq t_0$ in the domain of u . Then,

$$-v'(t) = \lambda u(t) + g(u(t)) \geq \min_{[d'_3, L'_0]} g := \delta' > 0.$$

So,

$$t_0^{(4)} - t_0 < \frac{v(t_0) - d'_1}{\delta'},$$

and $v(t_0^{(4)}) = d'_1$.

Case 5'. We define

$$d'_4 := \max\left\{d'_1, \frac{2d'_3}{T}\right\} + \nu'_{d'_3}T + \|E\|_\infty,$$

and assume

$$(u(t_0), v(t_0)) \in (-d'_1, d'_3) \times [d'_4, +\infty).$$

Then, there is a $t_0^{(5)} > t_0$ such that $u(t_0^{(5)}) = d'_3$ and, for every $t \in [t_0, t_0^{(5)})$,

$$(u(t), v(t)) \in (-d'_1, d'_3) \times (d'_1, +\infty).$$

Moreover,

$$t_0^{(5)} - t_0 \leq \frac{2d'_3}{v(t_0) - \nu'_{d'_3}T - \|E\|_\infty}.$$

In this way, Lemma 4.2 is proved under the assumptions of Theorem 2.3. The proof of Lemma 4.3 is exactly the same as in Section 4.

The proof of Lemma 4.4 follows from the fact that condition (h_1) together with (2.3) permits to have a lower estimate for the time-map (see [27]).

Lemma 4.5 is proved in the same way as in Section 4. Anyway, we remark that global existence for $(2.4)_\lambda$ can be proved under conditions (h_2) and (h_3) alone (cf. [23]).

The proof of Theorem 2.3 is now concluded like that of Theorem 2.1.

6. Related results and final remarks

Small modifications of the proofs above can be made in order to deal with an equation of the type

$$u''(t) + \mu u^+ - \nu u^- + g(t, u(t)) = e(t), \quad (6.1)_{\mu, \nu}$$

where μ and ν are positive parameters, and $e(t)$ and $g(t, u)$ satisfy the regularity assumptions of Section 2. The following results can then be proved.

Theorem 6.1. *Assume (i_0) and the following conditions:*

$$\lim_{|u| \rightarrow \infty} \frac{g(t, u)}{u} = 0,$$

uniformly for almost every $t \in [0, T]$;

$$\int_0^T \limsup_{u \rightarrow -\infty} g(t, u) dt < 0 < \int_0^T \liminf_{u \rightarrow +\infty} g(t, u) dt.$$

Then, there is an integer $k^* \geq 1$ such that, for every $k \geq k^*$ and $\mu > 0, \nu > 0$ satisfying

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} > kT,$$

(6.1) $_{\mu,\nu}$ has a periodic solution $u_{\mu,\nu,k}$ with minimal period kT having exactly two zeros in $[0, kT)$. For any sequence (u_n) of such solutions ($u_n = u_{\mu_n,\nu_n,k_n}$), if $k_n \rightarrow \infty$ (and hence $\mu_n \rightarrow 0$ and $\nu_n \rightarrow 0$) as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \min_{t \in \mathbb{R}} [|u_n(t)| + |u'_n(t)|] = +\infty.$$

Theorem 6.2. Assume (i_0) and that g does not depend on t , i.e., $g(t, u) = g(u)$. Let the following conditions hold:

$$\lim_{|u| \rightarrow \infty} \frac{G(u)}{u^2} = 0, \quad \exists d > 0, \quad |u| \geq d \Rightarrow g(u)u > 0, \quad \lim_{|u| \rightarrow \infty} G(u) = +\infty.$$

Then, the same conclusion of Theorem 6.1 holds.

Let us now consider an equation with a singularity. We may deal with

$$u''(t) + \lambda u + g(u(t)) = e(t), \tag{6.2}_\lambda$$

where $g : (0, +\infty) \rightarrow \mathbb{R}$, 0 being a singularity point. It is possible to prove the following result.

Theorem 6.3. Assume (i_0) and that g does not depend on t , i.e., $g(t, u) = g(u)$. Let the following conditions hold:

$$\lim_{u \rightarrow +\infty} \frac{G(u)}{u^2} = 0, \quad \exists d > 1, \quad u \in (0, d^{-1}) \cup (d, +\infty) \Rightarrow g(u)u > 0,$$

$$\lim_{u \rightarrow 0^+} G(u) = \lim_{u \rightarrow +\infty} G(u) = +\infty.$$

Then, there is an integer $k^* \geq 1$ such that, for every $k \geq k^*$ and $\lambda \geq 0$ satisfying

$$\lambda < \left(\frac{\pi}{kT} \right)^2,$$

(6.2) $_\lambda$ has a periodic solution $u_{\lambda,k}$ with minimal period kT such that $(u_{\lambda,k}(\cdot) - 1)$ has exactly two zeros in $[0, kT)$.

We have proved that the second-order differential equation (1.1), with $\delta = 0$, presents large-amplitude subharmonic solutions. We do not know whether these solutions survive or not when $\delta > 0$. Numerical simulation showed that they seem to survive for δ small. Since now, only variational methods or the Poincaré–Birkhoff theorem have been used to study subharmonic solutions in these situations. These methods preclude the study of equations with a friction term (see, however, [3]). It would be interesting to develop a method of proof of subharmonic solutions for such equations.

The equation considered is derived from a modification of our original model for a suspension bridge. The question naturally arises whether there exist subharmonic solutions to

the partial differential equation model, as well. Since now, we do not have any theoretical result along this direction. Numerical experiments are scheduled to be carried out.

Even if we are not in the position to claim that the oscillations responsible for the Tacoma Narrows Bridge disaster were of subharmonic type, this paper should at least produce the suspicion that the oscillations could have been of a nonlinear nature. The project of a very long bridge should also take into account the possibility of facing oscillations entering into the dangerous nonlinear region. We think that this kind of phenomena should be further investigated.

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