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An iterative method for the solvability of semilinear equations in Hilbert spaces and applications

1 An existence and uniqueness theorem

Let H be a Hilbert space. We are interested in the unique solvability of abstract equations of the form

$$Lu = Nu + h \tag{1}$$

where $L : D(L) \subset H \to H$ is a selfadjoint linear operator, $N : H \to H$ is a continuous gradient operator and $h \in H$ is arbitrary. We will provide some consequences and applications of the following abstract result, which we proved in [3].

Theorem 1. *Let $A, B : H \to H$ be two linear, continuous selfadjoint operators such that the following conditions hold.*

(i) $N - A$ and $B - N$ are monotone.
(ii) $L - (1-\mu)A - \mu B$ has a bounded inverse for every $\mu \in [0,1]$.

Then equation (1) has for every $h \in H$ a unique solution $u \in D(L)$ which can be obtained through the iterative process defined by

$$Lu_{k+1} - (1/2)(A+B)u_{k+1} = Nu_k - (1/2)(A+B)u_k + h,$$

with u_0 arbitrary in H.

As pointed out in [3], there is a quite vast literature on this problem. Our approach seems to be rather natural and easy, since a change of variable simply reduces equation (1) to a fixed point problem for a contraction mapping. For the reader's convenience, we reproduce here the proof of Theorem 1.

Proof. By condition (i), $B - A$ is nonnegative. As the set of operators with a bounded inverse is open, we can find, by condition (ii), an $\epsilon > 0$ such that $L-(1-\mu)(A-\epsilon I)-\mu(B+\epsilon I)$ has a bounded inverse for every $\mu \in [0,1]$. The operator $S = B - A + 2\epsilon I$ is selfadjoint, positive and invertible. Let $S^{1/2}$ and $S^{-1/2}$ be the square roots of S and S^{-1} respectively. We now operate the change of variable $v = S^{1/2}u$. Setting

$$\tilde{L} = S^{-1/2}(L - A + \epsilon I)S^{-1/2}, \ \tilde{N} = S^{-1/2}(N - A + \epsilon I)S^{-1/2},$$

equation (1) becomes equivalent to the equation

$$\tilde{L}v = \tilde{N}v + S^{-1/2}h, \tag{2}$$

where $v \in D(\tilde{L}) = S^{1/2}(D(L))$. Conditions (i) and (ii) become respectively

(i)' $\tilde{N}$ and $I - \tilde{N}$ are monotone;
(ii)' $\sigma(\tilde{L}) \cap [0,1] = \emptyset$,

where $\sigma(\tilde{L})$ denotes the spectrum of $\tilde{L}$. Furthermore, equation (2) is equivalent to the fixed point problem

$$v = (\tilde{L} - (1/2)I)^{-1}[\tilde{N}v - (1/2)v + S^{-1/2}h] := Tv.$$

By (i)' and Lemma 1 in [7], one has that $\tilde{N} - (1/2)I$ is Lipschitz continuous with Lipschitz constant 1/2. By (ii)' and the selfadjointness of $\tilde{L}$, one has

$$\|(\tilde{L} - (1/2)I)^{-1}\| < 2.$$

Consequently, T is a contraction, and the conclusion is a straightforward consequence of the fixed point theorem for contractive mappings. ∎

2 More explicit assumptions

Condition (ii) in Theorem 1 can be written more explicitely in case L commutes with A and B. Since this happens in many applications, we analyze this case below. This is based upon the following observation.

Proposition 1. *Let $D : H \to H$ be a continuous linear selfadjoint operator which commutes with L. If $\sigma(L) \cap \sigma(D) = \emptyset$, then $L - D$ has a bounded inverse.*

Proof. Since D and L commute, their spectral families commute ([9], ch. 8), and it is possible to find a spectral family $E(\lambda, \lambda')$ on the product space so that

$$L - D = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\lambda - \lambda')\, dE(\lambda, \lambda'),$$

(see [8], ch. 9). Since the spectra $\sigma(L)$ and $\sigma(D)$ are a positive distance apart, 0 cannot be in the spectrum of $L - D$. ∎

Remark that, in general, the two propositions "$\sigma(L) \cap \sigma(D) = \emptyset$" and "$L - D$ has a bounded inverse" do not imply each other ([3]). Moreover, even when L and D commute, the propositions are not equivalent, as shown by the following example in $H = \mathbf{R}^2$:

$$L(x_1, x_2) = (2x_1, x_2), \; D(x_1, x_2) = (x_1, 0).$$

Corollary 1. *Let A, B be two continuous linear selfadjoint operators which commute with L and are such that*

(i) $N - A$ and $B - N$ are monotone;
(ii) $\sigma(L) \cap \sigma[(1-\mu)A + \mu B] = \emptyset$ for every $\mu \in [0,1]$.

Then the conclusion of Theorem 1 holds.

Having in mind applications to systems of differential equations, we consider now the case where $H = E^m$, with E a Hilbert space.

Corollary 2. *Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric $m \times m$ matrices with respective eigenvalues $\alpha_1 \leq \ldots \leq \alpha_m$ and $\beta_1 \leq \ldots \leq \beta_m$. Denote by $A, B : E^m \to E^m$ the induced "constant" operators, and assume that they commute with L. If the following conditions are satisfied.*

(i) $N - A$ and $B - N$ are monotone.
(ii) $\sigma(L) \cap \bigcup_{i=1}^m [\alpha_i, \beta_i] = \emptyset$.

Then the conclusion of Theorem 1 holds.

The above result includes the ones of Amann [1], Theorem 3.2, and Dancer [2], Theorem 3. It is a straightforward consequence of Corollary 1.

3 Applications to differential equations

We want to give now applications to some boundary value problems for ordinary and partial differential equations. To this aim, let Ω be a measurable subset of $\mathbf{R}^n$, and $H = [L^2(\Omega)]^m$. Our first applications will deal with "diagonal" operators of a particular form, which will commute with every "constant" operator.

Corollary 3. *Assume that L is of the form $L(u_1, \ldots, u_m) = (\Lambda u_1, \ldots, \Lambda u_m)$, where $\Lambda : D(\Lambda) \subset L^2(\Omega) \to L^2(\Omega)$ is a selfadjoint operator. Let $G : \mathbf{R}^m \to \mathbf{R}$ be a differentiable function with gradient ∇G, and $\mathbf{A}, \mathbf{B}$ be two symmetric $m \times m$ matrices with eigenvalues $\alpha_1 \leq \ldots \leq \alpha_m$ and $\beta_1 \leq \ldots \leq \beta_m$, respectively. Assume that the following conditions hold.*

(j) $\langle \mathbf{A}(v_1 - v_2), v_1 - v_2 \rangle \leq \langle \nabla G(v_1) - \nabla G(v_2), v_1 - v_2 \rangle \leq \langle \mathbf{B}(v_1 - v_2), v_1 - v_2 \rangle$,
for every $v_1, v_2 \in \mathbf{R}^m$.
(jj) $\sigma(\Lambda) \cap \bigcup_{i=1}^m [\alpha_i, \beta_i] = \emptyset$.

Then the equation

$$(Lu)(t) = \nabla G(u(t)) + h(t)$$

has, for every $h \in [L^2(\Omega)]^m$, a unique solution $u \in D(L)$ which can be obtained, from any $u_0 \in [L^2(\Omega)]^m$, through the iterative process defined by

$$(Lu_{k+1})(t) - (1/2)(\mathbf{A} + \mathbf{B})u_{k+1}(t)$$
$$= \nabla G(u_k(t)) - (1/2)(\mathbf{A} + \mathbf{B})u_k(t) + h(t).$$

When dealing with differential equations, the above operators L and Λ are generally both denoted by Λ. Notice that for G twice differentiable, condition (j) of Corollary 3 is equivalent to the condition

(j)' $\mathbf{A} \leq G''(v) \leq \mathbf{B}$, for every $v \in \mathbf{R}^m$,

which was first introduced by Lazer [6].

Here are some examples of application for Corollary 3.

a) *Ordinary differential equations*

$$-u''(t) = \nabla G(u(t)) + h(t), \text{ in }]0,T[,$$
$$u(0) = u(T), \ u'(0) = u'(T).$$

In this case, $\sigma(\Lambda) = \{(kT/2\pi)^2 : k \in \mathbf{N}\}$.

b) *Elliptic problems*

$$-\Delta u(x) = \nabla G(u(x)) + h(x), \text{ in } \Omega,$$
$$u = 0 \text{ or } \partial u/\partial \nu = 0 \text{ on } \partial\Omega.$$

If Ω is a bounded domain, $\sigma(\Lambda)$ is made of eigenvalues with do not accumulate at any finite point. More general elliptic operators can be considered as well.

c) *Hyperbolic problems*

$$\Box u(t,x) = \nabla G(u(t,x)) + h(t,x) \text{ in }]0,T[\times\mathcal{D},$$
$$u(0,x) = u(T,x), \ u_t(0,x) = u_t(T,x) \text{ for } x \in \mathcal{D},$$
$$u(t,x) = 0 \text{ in }]0,T[\times\partial\mathcal{D}.$$

When $\mathcal{D}$ is a bounded domain, $\sigma(\Lambda)$ is the closure of the set obtained by making all the possible differences between the eigenvalues $(-kT/2\pi)^2$ of the corresponding periodic problem and those of the corresponding Dirichlet problem on $\mathcal{D}$. When $\mathcal{D}$ is the cube $]0,\pi[^n$, one has

$$\sigma(\Lambda) = \{m_1^2 + \ldots + m_n^2 - (kT/2\pi)^2 : m_i \in \mathbf{N}^*, k \in \mathbf{N}, i = 1,\ldots,n\}.$$

Other types of boundary conditions can be treated similarly, as e.g. periodic-Neumann, Dirichlet-Dirichlet, Dirichlet-Neumann.

d) *Schrödinger equations*

$$-\Delta u(x) + V(x)u(x) = \nabla G(u(x)) + h(x) \text{ in } \mathbf{R}^n, \ u \in [L^2(\mathbf{R}^n)]^m.$$

Here $V : \mathbf{R}^n \to \mathbf{R}$ is the potential. Corollary 3 applies once sufficient conditions can be given on V in order to guarantee the existence of gaps in the spectrum of $\Lambda = -\Delta + V$. We don't need extra technical conditions as in [1].

We now turn our attention toward operators which do not satisfy the commutativity properties of Corollaries 1 to 3. In those cases, we will apply directly Theorem 1.

e) *Hamiltonian systems*

Consider the Hamiltonian system

$$Ju'(t) = \nabla G(u(t)) + h(t), \text{ in }]0,T[,$$
$$u(0) = u(T),$$

where $m = 2l$, $J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ is the symplectic matrix. Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric $m\times m$ matrices, and let $A, B : [L^2(0,T)]^m \to [L^2(0,T)]^m$ be the induced 'constant' operators. Notice that the operator $L = J\frac{d}{dt}$ does not commute in general with A and B. Assume for instance that condition

(h) $\mathbf{A} \le G''(v) \le \mathbf{B}$, $(v \in \mathbf{R}^n)$

holds. In order to apply Theorem 1, we need to check its assumption (ii), i.e., as L has compact resolvent, that the problem

$$Ju' + (1-\mu)\mathbf{A}u + \mu\mathbf{B}u = 0,\ u(0) = u(T),$$

has only the trivial solution for every $\mu \in [0,1]$. It is easy to check that this is equivalent to the condition

(hh) $\bigcup_{\mu\in[0,1]} \sigma[(1-\mu)J\mathbf{A} + \mu J\mathbf{B}] \cap \frac{2\pi}{T}i\mathbf{Z} = \emptyset$.

We can conclude that, if conditions (h) and (hh) are satisfied, the Hamiltonian system above has a unique T-periodic solution for every $h \in [L^2(0,T)]^m$, and the iterative process of Theorem 1 converges to the solution. We can remark that condition (hh) generalizes the one found when dealing with second order systems. Moreover, condition (hh) can be shown to be equivalent to the following one, introduced by Amann [1], via a finite dimensional reduction.

Condition (hh)'. *Suppose that*

$$\sigma(J\mathbf{A}) \cap \frac{2\pi}{T}i\mathbf{Z} = \emptyset = \sigma(J\mathbf{B}) \cap \frac{2\pi}{T}i\mathbf{Z},$$

and choose $\beta \in \mathbf{R}_+ \setminus \frac{2\pi}{T}\mathbf{Z}$ *such that* $-\beta I \le \mathbf{A} \le \mathbf{B} \le \beta I$. *Denote by E the sum of the eigenspaces of L belonging to the eigenvalues in* $]-\beta,\beta[$, *and assume, with m the Morse index, that* $m((L-A)|_E) = m((L-B)|_E)$.

f) *Dirac equations*

Consider the problem of finding $u \in L^2(\mathbf{R}^3, \mathbf{C}^4)$ such that

$$\sum_{j=1}^{3}[\alpha_j \frac{\partial u}{\partial x_j}(x) - Q_j(x)\alpha_j u(x)] + m\alpha_4 u(x) = \nabla G(u(x)) + h(x),$$

where $Q = (Q_1, Q_2, Q_3) : \mathbf{R}^3 \to \mathbf{R}^3$ is the magnetic potential, $m > 0$, and the α_j are the 4×4 matrices defined by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \ (1 \le j \le 3), \ \ \alpha_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where the σ_j are the Pauli matrices satisfying the relations

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}, \ \sigma_1\sigma_2 = i\sigma_3, \ ; \sigma_2\sigma_3 = i\sigma_1, \ \sigma_3\sigma_1 = i\sigma_2.$$

One can take for instance,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known that the spectrum $\sigma(L)$ of the Dirac operator defined above is such that

$$\sigma(L) \cap]-m, m[= \emptyset,$$

(see e.g. [4] and [5]). Assume that, for some $\rho \in [0, m[$ and $a_j, b_j \in \mathbf{R}$, $(j = 1, 2, 3)$, we have

$$-\rho I_4 - \sum_{j=1}^{3} a_j\alpha_j \le G''(v) \le \rho I_4 + \sum_{j=1}^{3} b_j\alpha_j,$$

for every $v \in \mathbf{C}^4$. The, by Theorem 1, the problem has a unique solution for every $h \in L^2(\mathbf{R}^3, \mathbf{C}^4)$. In fact, condition (ii) of Theorem 1 is verified since, for every $\mu \in [0, 1]$, the kernel of the operator

$$\sum_{j=1}^{3}[\alpha_j \frac{\partial}{\partial x_j} - Q_j\alpha_j] + m\alpha_4 - (1-\mu)[-\rho I_4 - \sum_{j=1}^{3} a_j\alpha_j] - \mu[\rho I_4 + \sum_{j=1}^{3} b_j\alpha_j] =$$

$$\sum_{j=1}^{3}[\alpha_j \frac{\partial}{\partial x_j} - (Q_j - (1-\mu)a_j + \mu b_j)\alpha_j] + m\alpha_4 + (1-2\mu)\rho I_4$$

is trivial, as $(1-2\mu)\rho \in]-m, m[$.

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