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Quadratic forms, weighted eigenfunctions and boundary value problems for non-linear second order ordinary differential equations

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Synopsis

Some known results for different kinds of boundary value problems for second order ordinary differential equations are generalised. Different approaches are compared with one another, using topological and variational methods and the theory of weighted eigenvalue problems.

1. Introduction

This paper is devoted to the study of various auxiliary tools employed when dealing with boundary value problems associated with second order differential equations of the form

$$x'' + f(t, x) = 0. \tag{1.1}$$

Many existence conditions for equation (1.1) deal with the relation between the asymptotic behaviour of the non-linearity f and the spectrum of the differential operator. In particular, Mawhin and Ward (cf. **[13, 15, 16]**) have introduced and used some quadratic forms associated with the eigenvalues and eigenfunctions of $-x''$. In this way they were able to treat many cases where $f(t, x)/x$ stays asymptotically between two consecutive eigenvalues or to the left of the spectrum.

On the other hand, Lasota and Opial **[11]** introduced a method of study of some boundary value problems for equation (1.1) which gives the existence of solutions in particular when $f(t, x)/x$ behaves asymptotically as a function $p(t)$, such that the equation

$$x''(t) + p(t)x(t) = 0, \tag{1.2}$$

with the corresponding boundary conditions, has only the trivial solution. This approach was recently extended by Habets and Metzen **[10]** to the case of a jumping non-linearity.

It is natural to study the relations between the two approaches. The aim of this paper is to provide a complete comparison as well as some abstract theorems generalising the above mentioned results.

Section 2 emphasises the fact that many known existence results for some boundary value problems associated with equation (1.1), when $f(t,x)/x$ stays asymptotically at the left of the spectrum of $-x''$, or between two consecutive eigenvalues, still hold when we only assume that some quadratic forms, naturally associated with the asymptotic behaviour of $f(t,x)/x$, are positive definite. This observation, combined with topological or variational approaches, provides some general existence theorems.

In Section 3 we prove the equivalence of the approach based on the quadratic forms, and that based on the above property of the associated linear problem (1.2), in the case of the Dirichlet or Neumann conditions. This is done in a straightforward way by using the weighted regular Sturm–Liouville theory as developed in [**1–3**]. The extension of this linear theory to the periodic case would make it possible to prove the analogues of Theorems 3.1 and 3.2 under periodic boundary conditions.

2. Existence results by the use of quadratic forms

We consider the following second order differential equation:

$$x''+f(t,x)=0,$$

where $f: I\times\mathbb{R}\to\mathbb{R}$ $(I=[0,T])$ is a Carathéodory function, i.e. $f(\cdot,x)$ is measurable for every $x\in\mathbb{R}$ and $f(t,\cdot)$ is continuous for almost every $t\in I$. Moreover, we assume the following condition.

CONDITION 2.1. For every $R>0$ there is a $k_R\in L^1(I)$ such that

$$|f(t,x)|\leqq k_R(t),$$

for all $|x|\leqq R$ and almost every $t\in I$.

Associated with equation (1.1) we consider one of the following boundary conditions:

the Dirichlet conditions $x(0)=x(T)=0$;
the Neumann conditions $x'(0)=x'(T)=0$;
the periodic conditions $x(0)-x(T)=x'(0)-x'(T)=0$.

We will look for C^1-functions x with absolutely continuous derivatives (i.e. $x\in W^{2,1}(I)$) verifying equation (1.1) almost everywhere and one of the above boundary conditions. Such a function x will be called a solution of the considered boundary value problem.

According to which of the boundary conditions is considered, we will denote by H^1_* and W_* the following sets:

Dirichlet problem:

$$H^1_*=\{x\in H^1(I)\mid x(0)=x(T)=0\},$$
$$W_*=\{x\in W^{2,1}(I)\mid x(0)=x(T)=0\};$$

Neumann problem:

$$H^1_* = H^1(I),$$
$$W_* = \{x \in W^{2,1}(I) \mid x'(0) = x'(T) = 0\};$$

periodic problem:

$$H^1_* = \{x \in H^1(I) \mid x(0) = x(T)\},$$
$$W_* = \{x \in W^{2,1}(I) \mid x(0) - x(T) = x'(0) - x'(T) = 0\}.$$

Our first existence result is the following generalisation of a result in **[13]**.

THEOREM 2.2. *Let* $b \in L^1(I)$ *be such that*:

(A1) *for every* $\varepsilon > 0$ *there exist* $\beta_\varepsilon, \gamma_\varepsilon \in L^1(I)$ *such that*

$$f(t, x)x \leqq (b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)\,|x| + \gamma_\varepsilon(t);$$

(B1) *for any* $x \in H^1_* \setminus \{0\}$ *one has* $\int_I ((x')^2 - bx^2) > 0$. *Then* (1.1) *has a solution in* W_*.

To prove Theorem 2.2 we need the following lemma, which is essentially proved in **[13]**.

LEMMA 2.3. *Condition* (B1) *is equivalent to*

(B2) *there exists* $\bar{\varepsilon} > 0$ *such that for any* $x \in H^1_*$ *one has*

$$\int_I ((x')^2 - bx^2) \geqq \bar{\varepsilon}\, \|x\|^2_{H^1}.$$

Proof. If (B2) is false, we can find a sequence (x_n) in H^1_* such that $\|x_n\|_{H^1} = 1$ and $\int_I ((x_n')^2 - bx_n^2) \to 0$. Taking a subsequence, we can assume $x_n \rightharpoonup x$ in $H^1(I)$. Then (x_n) converges uniformly, and the weak lower semicontinuity of the L^2-norm of x_n' implies $\int_I ((x')^2 - bx^2) \leqq 0$. By (B1), $x = 0$, and the above implies that (x_n) converges uniformly towards zero. Since $\int_I ((x_n')^2 - bx_n^2) \to 0$, it follows that $\|x_n\|_{H^1} \to 0$, which is impossible.

Proof of Theorem 2.2. Let us define the following operators:

$$\operatorname{dom}(\mathscr{L}) = W_*$$
$$\mathscr{L}: \operatorname{dom}(\mathscr{L}) \to L^1(I),\ x \to x'',$$
$$N: C(I) \to L^1(I), \quad x \to f(\cdot, x(\cdot)).$$

It is well known (cf. **[7, 12]**) that N is $\mathscr{L}$-completely continuous, and the result will be proved if we find an *a priori* bound for the solutions in W_* of the equations

$$x'' + \lambda f(t, x) + (1 - \lambda)b(t)x = 0 \tag{2.1}$$

for every $\lambda \in [0, 1]$. With this aim, fix $\varepsilon < \bar{\varepsilon}$, multiply equations (2.1) by $-x$ and integrate, to obtain

$$0 = \int_I (x')^2 - \lambda f(t, x)x - (1 - \lambda)b(t)x^2$$
$$\geqq \int_I (x')^2 - \lambda[(b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)\,|x| + \gamma_\varepsilon(t)] - (1 - \lambda)b(t)x^2$$
$$\geqq (\bar{\varepsilon} - \varepsilon)\, \|x\|^2_{H^1} - C\, \|\beta_\varepsilon\|_{L^1}\, \|x\|_{H^1} - \|\gamma_\varepsilon\|_{L^1}.$$

The *a priori* bound follows.

Let us define the function F by

$$F(t, x) = \int_0^x f(t, s)\, ds$$

Then we have a result which is the analogue of Theorem 2.2 and extends [**14**, Theorem 1.1].

THEOREM 2.4. *Let $b \in L^1(I)$ be such that*
(A2) *for every $\varepsilon > 0$ there exist β_ε, $\gamma_\varepsilon \in L^1(I)$ such that*

$$2F(t, x) \leqq (b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)\,|x| + \gamma_\varepsilon(t),$$

and suppose (B1) *holds. Then equation* (1.1) *has a solution in W_*.*

Proof. We consider the functional associated with our problem, defined on H^1_*:

$$\rho(x) = \int_I [(x'(t))^2 - 2F(t, x(t))]\, dt. \tag{2.2}$$

One can easily see that ρ is weakly lower semicontinuous. Moreover, fix $\varepsilon < \bar{\varepsilon}$. By Lemma 2.3,

$$\begin{aligned}\rho(x) &\geqq \int_I \{(x')^2 - [(b(t) + \varepsilon)x^2 + \beta_\varepsilon(t)\,|x| + \gamma_\varepsilon(t)]\}\\ &\geqq (\bar{\varepsilon} - \varepsilon)\,\|x\|^2_{H^1} - c\,\|\beta_\varepsilon\|_{L^1}\,\|x\|_{H^1} - \|\gamma_\varepsilon\|_{L^1}.\end{aligned}$$

Hence ρ is coercive, and then it has a minimum, giving the solution we are looking for.

Remarks 2.5. (1) Theorems 2.2 and 2.4 can be extended in a straightforward way to the vector case. In Theorem 2.2 the products must be replaced by scalar products in $\mathbb{R}^n$ and for Theorem 2.4 the system must be supposed to be in variational form, i.e. $f = D_x F$ for some $F: I \times \mathbb{R}^n \to \mathbb{R}$.

(2) Using Condition 2.1 it is easy to show that assumption (A1) is verified if, for some $b \in L^1(I)$, the following holds uniformly for almost every $t \in I$.

$$\limsup_{|x| \to \infty} f(t, x)/x \leqq b(t).$$

Analogously, assumption (A2) is verified if, uniformly for almost every $t \in I$,

$$\limsup_{|x| \to \infty} 2F(t, x)/x^2 \leqq b(t).$$

(3) Sufficient conditions for (B1) to hold can be found in [**4, 8, 9, 13, 15, 16**].

We will now introduce a condition (B3) which generalises (B1) and prove analogous existence results.

We begin with the following lemma, which is an immediate consequence of Condition 2.1.

PROPOSITION 2.6. *Assume a, $b \in L^1(I)$ are such that*

(A3) $a(t) \leqq \liminf_{|x| \to \infty} f(t, x)/x \leqq \limsup_{|x| \to \infty} f(t, x)/x \leqq b(t).$

Then the following holds :

(A4) *for every* $\varepsilon > 0$ *there exist* $g_\varepsilon, h_\varepsilon: I \times \mathbb{R} \to \mathbb{R}$, $\check{h}_\varepsilon \in L^1(I)$ *such that*

$$f(t, x) = g_\varepsilon(t, x)x + h_\varepsilon(t, x)$$
$$a(t) - \varepsilon \leqq g_\varepsilon(t, x) \leqq b(t) + \varepsilon$$
$$|h_\varepsilon(t, x)| \leqq \check{h}_\varepsilon(t).$$

The following theorem is a generalisation of a result in **[13]**.

THEOREM 2.7. *Let* $a, b \in L^1(I)$ *satisfy* (A4) *and*

(B3) $L^2(I) = H^- \oplus H^+$, *where* H^- *is finite dimensional and contained in* H^1_*, *and for every* $x = \bar{x} + \check{x} \in H^1_* \backslash \{0\}$, *with* $\bar{x} \in H^-$ *and* $\check{x} \in H^+$, *one has*

$$B_{a,b}(x) := \int_I ((\check{x}')^2 - b\check{x}^2) - \int_I ((\bar{x}')^2 - a\bar{x}^2) > 0.$$

Then (1.1) *has a solution in* W_*.

To prove Theorem 2.7 we need the following lemma, which is essentially proved in **[13]**.

LEMMA 2.8. *If* (B3) *holds, then there exists* $\bar{\varepsilon} > 0$ *such that, for every* $x \in H^1_*$,

(B4) $B_{a,b}(x) \geqq \bar{\varepsilon}\, \|x\|^2_{H^1}$.

Proof. If the conclusion is false, one can find a sequence (x_n) in H^1_* such that $\|x_n\|_{H^1} = 1$ and $B_{a,b}(x_n) \to 0$. Taking a subsequence, we can suppose $x_n \rightharpoonup x$ in $H^1(I)$. Then $\bar{x}_n \to \bar{x}$ in $H^1(I)$, all norms being equivalent in a finite dimensional space, and also $\check{x}_n \to \check{x}$ uniformly. So,

$$\int_I (\check{x}_n')^2 \to \int_I (b\check{x}^2 + (\bar{x}')^2 - a\bar{x}^2),$$

and by the weak lower semicontinuity of the L^2-norm of $\check{x}_n'$ we obtain $B_{a,b}(x) \leqq 0$. Then, by (B3), $x = 0$. It then follows from the above that $\|x_n\|_{H^1} \to 0$, which is a contradiction. □

Proof of Theorem 2.7. Fix $\varepsilon < \bar{\varepsilon}$. By the arguments in the proof of Theorem 2.2, the proof will be complete if we find an *a priori* bound for the solutions of equations (2.1) in W_* for every $\lambda \in [0, 1]$. By multiplying by $(\bar{x} - \check{x})$ and integrating, we obtain

$$0 = \int_I [(\check{x}')^2 - (\bar{x}')^2 + \lambda g_\varepsilon(t, x)(\bar{x}^2 - \check{x}^2) + \lambda h_\varepsilon(t, x)(\bar{x} - \check{x}) + (1 - \lambda)b(t)(\bar{x}^2 - \check{x}^2)]$$
$$\geqq \int_I [(\check{x}')^2 - (\bar{x}')^2 + (a(t) - \varepsilon)\bar{x}^2 - (b(t) + \varepsilon)\check{x}^2 + \lambda h_\varepsilon(t, x)(\bar{x} - \check{x})]$$
$$\geqq (\bar{\varepsilon} - \varepsilon)\, \|x\|^2_{H^1} - c\, \|\check{h}_\varepsilon\|_{L^1}\, \|x\|_{H^1},$$

and the *a priori* bound follows. □

A similar result can be obtained by a variational method if assumption (A4) is replaced by an analogous one concerning the primitive of f. In this case we have to consider the functional ρ defined as in equations (2.1), whose critical points are precisely the solutions of the boundary value problem associated with equation (1.1). Let us recall the definition of the Palais–Smale condition (PS).

DEFINITION 2.9. The functional ρ satisfies (PS) if any sequence (x_n) in H^1_* such that $\rho(x_n)$ is bounded and $\rho'(x_n) \to 0$ has a convergent subsequence.

PROPOSITION 2.10. *Assume a, $b \in L^1(I)$ are such that*

(A5) $a(t) \leqq \liminf_{|x| \to \infty} 2F(t, x)/x^2 \leqq \limsup_{|x| \to \infty} 2F(t, x)/x^2 \leqq b(t)$.

Then the following holds:
(A6) *for every $\varepsilon > 0$ there exist G_ε, $H_\varepsilon : I \times \mathbb{R} \to \mathbb{R}$, $\mathcal{H}_\varepsilon \in L^1(I)$ such that*

$$2F(t, x) = G_\varepsilon(t, x)x^2 + H_\varepsilon(t, x)$$
$$a(t) - \varepsilon \leqq G_\varepsilon(t, x) \leqq b(t) + \varepsilon$$
$$|H_\varepsilon(t, x)| \leqq \mathcal{H}_\varepsilon(t).$$

THEOREM 2.11. *Let a, $b \in L^1(I)$ satisfy (B3) and (A6). If, moreover, the functional ρ satisfies (PS), then equation (1.1) has a solution in W_*.*

Proof. Take in (A6) $\varepsilon = \bar{\varepsilon}/2$, with $\bar{\varepsilon}$ as in Lemma 2.8. If $\bar{x} \in H^-$, one has, by (A6) and (B4),

$$\begin{aligned}\rho(\bar{x}) &= \int_I [(\bar{x}'(t))^2 - G_\varepsilon(t, \bar{x}(t))\bar{x}^2 - H_\varepsilon(t, \bar{x}(t))]\, dt \\ &\leqq \int_I [(\bar{x}')^2 - (a(t) - \varepsilon)\bar{x}^2] + \|\mathcal{H}_\varepsilon\|_{L^1} \\ &= -B_{a,b}(\bar{x}) + \varepsilon\, \|\bar{x}\|^2_{L^2} + C \\ &\leqq -(\bar{\varepsilon}/2)\, \|\bar{x}\|^2_{H^1} + C,\end{aligned}$$

where $C = \|\mathcal{H}_\varepsilon\|_{L^1}$. Analogously, if $\tilde{x} \in H^+$, one proves that

$$\rho(\tilde{x}) \geqq (\bar{\varepsilon}/2)\, \|\tilde{x}\|^2_{H^1} - C.$$

The above implies that we are in the geometrical setting of the Saddle point theorem of Rabinowitz (cf. [**17**]), and since ρ satisfies (PS), the result follows. □

Remark 2.12. Conditions which imply (B3) are given in [**4**, **8**, **9**, **13**, **15**, **16**]. Some sufficient conditions for (PS) to hold can be found in [**14**, **17**].

3. A coercive quadratic form for some classes of non-coercive linear problems

In this section we will restrict ourself to Dirichlet or Neumann boundary conditions.

THEOREM 3.1. *Given $b \in L^1(I)$, the following assertions are equivalent*:
(i) *assumption* (B1) *holds*;
(ii) *for each $p \in L^1(I)$ such that $p(t) \leqq b(t)$ for almost every $t \in I$, the equation*

$$x'' + p(t)x = 0 \qquad (3.1)$$

has only the trivial solution in W_*;

(iii) *for every* $\lambda \leqq 0$ *the equation*

$$x'' + (b(t) + \lambda)x = 0 \tag{3.2}$$

has only the trivial solution in W_*.

Proof. To show that (i) implies (ii), take p as in (ii) and let x be a solution of equation (3.1) in W_*. Multiply equation (3.1) by $(-x)$ and integrate, to obtain

$$0 = \int_I ((x')^2 - px^2) \geqq \int_I ((x')^2 - bx^2).$$

It then follows from (B1) that $x = 0$.

It is clear that (ii) implies (iii). Let us then show that (iii) implies (i). By the theory of linear second order differential operators (see [**1–3**]), the eigenvalues of equation (3.2) with Dirichlet or Neumann boundary conditions form a sequence $\lambda_1 < \lambda_2 < \cdots$ which tends to $+\infty$, and the corresponding eigenfunctions ϕ_1, $\phi_2, \ldots$ are an orthonormal base of $L^2(I)$. Hence, given any $x \in H^1_*$, we can write

$$x(t) = \sum_{i \geqq 1} c_i \phi_i(t),$$

and

$$\int_I ((x')^2 - bx^2) = \sum_{i \geqq 1} c_i^2 \int_I ((\phi_i')^2 - b\phi_i^2)$$

$$= \sum_{i \geqq 1} c_i^2 \lambda_i \int_I \phi_i^2$$

$$\geqq \lambda_1 \int_I x^2.$$

By (iii), $\lambda_1 > 0$, and (B1) follows. □

An analogous result holds when (B3) is considered. Given a, $b \in L^1(I)$ as in (B3), we can suppose without restriction that $a(t) < b(t)$ for almost every $t \in I$. In fact, by Lemma 2.8, if a, b verify condition (B3), this is also true for $a - \bar{\varepsilon}/4$, $b + \bar{\varepsilon}/4$.

THEOREM 3.2. *Given* a, $b \in L^1(I)$ *such that* $a(t) < b(t)$ *for almost every* $t \in I$, *the following assertions are equivalent:*

(i) *assumption* (B3) *holds*;

(ii) *taking* $p \in L^1(I)$ *such that* $a(t) \leqq p(t) \leqq b(t)$ *for almost every* $t \in I$, *the equation*

$$x'' + p(t)x = 0 \tag{3.3}$$

has only the trivial solution in W_*;

(iii) *for every* $\mu \in [0, 1]$, *the problem*

$$x'' + [(1 - \mu)a(t) + \mu b(t)]x = 0 \tag{3.4}$$

has only the trivial solution in W_*.

Proof. Assume (B3) holds. By multiplying equation (3.3) by $(\bar{x}-\check{x})$ and integrating, one has

$$0=\int_I ((\check{x}')^2-(\bar{x}')^2)+p(\bar{x}^2-\check{x}^2)\geqq B_{a,b}(x).$$

Hence $x=0$, so that (i) implies (ii).

It is clear that (ii) implies (iii). Let us show that (iii) implies (i). Notice that equation (3.3) can be written as

$$x''+ax+\mu(b-a)x=0. \tag{3.5}$$

The theory developed in **[1–3]** can be applied: the eigenvalues of equation (3.5) with Dirichlet or Neumann boundary conditions form a sequence $\mu_1<\mu_2<\dots$ which tends to $+\infty$. The corresponding eigenfunctions $\psi_1,\psi_2,\dots$ are an orthonormal base in the space $L^2_{b-a}(I)$ of measurable functions u such that

$$\int_I (b(t)-a(t))u(t)^2\,dt<+\infty,$$

with scalar product given by

$$(u\mid v)=\int_I (b(t)-a(t))u(t)v(t)\,dt.$$

Hence, for $i\neq j$, one has

$$\int_I (b-a)\psi_i\psi_j=0$$

and, as one can easily verify,

$$\int_I (\psi_i'\psi_j'-a\psi_i\psi_j)=\int_I (\psi_i'\psi_j'-b\psi_i\psi_j)=0. \tag{3.6}$$

Given $x\in H^1_*$, we can write $x(t)=\sum_{i\geqq 1}c_i\psi_i(t)$. By (iii), there are no eigenvalues μ_n in the interval $[0,1]$. So, either $\mu_1>1$, or there exists an $n\geqq 1$ such that $\mu_n<0<1<\mu_{n+1}$. If $\mu_1>1$, define $H^-=\{0\}$ and $H^+=L^2_{b-a}(I)$. Then, by equation (3.6) one has:

$$\begin{aligned}B_{a,b}(x)&=\int_I ((x')^2-bx^2)\\&=\sum_{i\geqq 1}c_i^2\int_I ((\psi_i')^2-b\psi_i^2)\\&=\sum_{i\geqq 1}c_i^2(\mu_i-1)\int_I ((b-a)\psi_i^2)\\&\geq(\mu_1-1)\int_I ((b-a)x^2),\end{aligned}$$

and (B3) follows. On the other hand, if $\mu_n<0<1<\mu_{n+1}$, set $H^-=\operatorname{span}\{\psi_1,\dots,\psi_n\}$ and $H^+=\operatorname{span}\{\psi_{n+1},\psi_{n+2},\dots\}$. With analogous calculations, one has

$$\int_I ((\tilde{x}')^2 - b\tilde{x}^2) \geqq (\mu_{n+1} - 1)\int_I ((b-a)\tilde{x}^2),$$

$$\int_I ((\bar{x}')^2 - a\bar{x}^2) \leqq \mu_n \int_I ((b-a)\bar{x}^2),$$

and (B3) follows in this case, as well.

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