Phase space analysis for wave models

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1 Wave models

We can characterize terms of lower order in wave equations. Let us begin with the classical wave equation

$$u_{tt} - \Delta u = 0.$$

This equation describes the propagation of waves. It appears in numerous models as for the vibrating string or membrane, the propagation of sound, the longitudinal vibrations of an elastic rod or beam, surface water waves, the propagation of electric signals or for the description of electric or magnetic fields.

Klein (1927) and Gordon (1926) derived the following Klein-Gordon equation describing a charged particle in an electro-magnetic field:

$$u_{tt} - \Delta u + m^2 u = 0.$$

The term m^2u is called *mass or potential*.

A well-known equation is the *telegraph equation*

$$u_{tt} - u_{xx} + au_t + bu = 0,$$

where a and b are constants. This equation arises in the study of propagation of electric signals in a cable of transmission line, in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge. Here bu is a *mass term*, and au_t is a *damping term* or *dissipation term*. A higher-dimensional generalization is

$$u_{tt} - \Delta u + au_t + bu = 0.$$

Finally, we mention wave equations with a *convection term* or a *transport term* $\sum_{k=1}^n a_k(t,x)\partial_{x_k}u$, that is,

$$u_{tt} - \Delta u + \sum_{k=1}^{n} a_k(t,x)\partial_{x_k} u = 0.$$

2 Basics of Fourier transformation

The Fourier transformation is a special integral transformation. Usually it is defined by (classical definition)

$$F(f)(\xi) := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

with $x \cdot \xi = \sum\limits_{l=1}^{n} x_l \xi_l$. The inverse Fourier transformation is defined by

$$F^{-1}(g)(x) := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)d\xi.$$

2.1 Application to spaces of infinitely differentiable functions

Let us choose the function space $C^\infty(\mathbb{R}^n)$. Functions f from this space could be unbounded for $|x| \to \infty$, for example, $f(x) = e^{|x|^4}$. Such an unbounded behavior does in general not imply the convergence of the above integrals for $F(f)$ or for $F^{-1}(g)$. Let us choose the space $C_0^\infty(\mathbb{R}^n)$. Then elements f have compact support. Consequently, the above integrals for $F(f)$ or for $F^{-1}(g)$ exist. We can not expect that $F(f)$ or $F^{-1}(g)$ belong to $C_0^\infty(\mathbb{R}^n)$ if f or g belong to. Why? But there exists a kind of intermediate space between $C^\infty(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ which has the property that f and g from this space imply $F(f)$ and $F^{-1}(g)$ from the same space. This is the so-called *Schwartz space* $S(\mathbb{R}^n)$, the *space of fast decreasing functions.*

Definition 2.1. *By $S(\mathbb{R}^n)$ we denote the subspace of $C^\infty(\mathbb{R}^n)$ consisting of all functions f which satisfy the conditions*

$$p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial_x^\alpha f(x) \right| < \infty$$

for all multi-indices α and β. The topology in $S(\mathbb{R}^n)$ is generated by the family of semi-norms $\{p_{\alpha,\beta}(f)\}_{\alpha,\beta}$.

The Schwartz space is the largest subspace of $L^1(\mathbb{R}^n)$ which is invariant with respect to the operations differentiation ∂_x^α and multiplication by x^β.

Theorem 2.1. *The Fourier transformation and the inverse Fourier transformation are mapping continuously the Schwartz space into itself. The Fourier transform of $\partial_{x_k} f$ is $i\xi_k F(f)$ and the Fourier transform of $x_k f$ is $i\partial_{\xi_k} F(f)$. In this way, a differentiation in the physical space, that is the space $\mathbb{R}^n_x$, corresponds to a multiplication by the phase space variable in the phase space, that is the space $\mathbb{R}^n_\xi$, and conversely.*

Proof. We restrict ourselves to show that for a given f from $S(\mathbb{R}^n)$ the image $F(f)$ belongs to $S(\mathbb{R}^n)$, too. Straight-forward calculations imply

$$
\begin{aligned}
&\xi^\beta \partial_\xi^\alpha F(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi}\xi^\beta(-ix)^\alpha f(x)dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} i^{|\beta|}\partial_x^\beta(e^{-ix\xi})(-ix)^\alpha f(x)dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi}(-i)^{|\beta|}\partial_x^\beta((-ix)^\alpha f(x))dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi}(1+|x|^2)^{-\frac{n+1}{2}}(-i)^{|\beta|}(1+|x|^2)^{\frac{n+1}{2}}\partial_x^\beta((-ix)^\alpha f(x))dx.
\end{aligned}
$$

Here the assumption $f \in S(\mathbb{R}^n)$ yields, that during partial integration in the above integrals boundary integrals are vanishing. Moreover, we have

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|^2)^{\frac{n+1}{2}}\partial_x^\beta((-ix)^\alpha f(x))\right| < \infty.$$

Taking account of $\int_{\mathbb{R}^n}(1+|x|^2)^{-\frac{n+1}{2}}dx < \infty$ gives in the phase space

$$p_{\alpha,\beta}(F(f)) = \sup_{\xi\in\mathbb{R}^n} \left|\xi^\beta\partial_\xi^\alpha F(f)(\xi)\right| < \infty$$

for all multi-indices α and β. In the same way we can show that $p_{\alpha,\beta}(f_k - f) \to 0$ for $k \to \infty$ implies $p_{\alpha,\beta}(F(f_k) - F(f)) \to 0$ for all multi-indices α and β. Consequently, $F: \ f \to F(f)$ maps the space $S(\mathbb{R}^n)$ continuously into itself. The rules are proved as above. □

Intuitively, we expect the following *Fourier inversion formula*:

$$F^{-1}(F(f))(x) = f(x).$$

This inversion formula holds for all functions $f \in S(\mathbb{R}^n)$. For proving this statement we need so-called *regularization*. For a given function $g \in C_0^\infty(\mathbb{R}^n)$, $g(x) \geq 0$, $\int_{\mathbb{R}^n} g(x)dx = 1$, we define the function $g_\varepsilon = \varepsilon^{-n}g\left(\frac{x}{\varepsilon}\right)$. Let $f \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$. Then we define the regularization $J_\varepsilon(f)$ of f by the aid of the convolution integral $J_\varepsilon(f) := g_\varepsilon * f := \int_{\mathbb{R}^n} g_\varepsilon(x-y)f(y)dy$. For every $\varepsilon > 0$ the regularization $J_\varepsilon(f)$ belongs to $C^\infty(\mathbb{R}^n)$. Such a regularization satisfies the following remarkable property:

$$\lim_{\varepsilon\to 0} \|J_\varepsilon(f) - f\|_{L^p(\mathbb{R}^n)} = \lim_{\varepsilon\to 0} \|g_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} = 0.$$

Remark 2.1. *The assumption $g \in C_0^\infty(\mathbb{R}^n)$ gives $J_\varepsilon(f) \in C^\infty(\mathbb{R}^n)$. But $J_\varepsilon(g) \in C^\infty(\mathbb{R}^n)$ holds for all functions $g \in L^1(\mathbb{R}^n)$.*

By Remark 2.1 we are able to prove the Fourier inversion formula for all functions $f \in S(\mathbb{R}^n)$. We introduce a function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(\eta) = \begin{cases} 1 & |\eta| \leq 1, \\ 0 & |\eta| \geq 2. \end{cases}$ Then we have

$$F^{-1}(F(f))(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{i(x-y)\cdot\xi} f(y) dy d\xi = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{i(x-y)\cdot\xi} f(y) \chi(\varepsilon\xi) dy d\xi$$
$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(y) \varepsilon^{-n} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{i\frac{x-y}{\varepsilon}\cdot\eta} \chi(\eta) d\eta dy = \lim_{\varepsilon \to 0} f * g_\varepsilon(x),$$

with the function $g = g(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\eta} \chi(\eta) d\eta \in L^1(\mathbb{R}^n)$. By Remark 2.1 we conclude the *Fourier inversion formula* $F^{-1}(F(f))(x) = f(x)$.

Example 2.1. *Important for $L^p - L^q$ estimates*
In applications there appear Fourier transforms for Gauß' functions. Let $(Ax, x) = \sum_{k,l=1} a_{kl} x_k x_l$ *be a positive definite quadratic form. Then we have*

$$F(e^{-(Ax,x)}) = \frac{1}{\sqrt{2}^n \sqrt{\det A}} e^{-\frac{1}{4}(\xi, A^{-1}\xi)}.$$

Special case: *Let us choose* $A = \frac{1}{2}I$. *Then*

$$F\Big(e^{-\frac{|x|^2}{2}}\Big) = \frac{1}{\sqrt{2}^n (\frac{1}{2})^{n/2}} e^{-\frac{|\xi|^2}{2}} = e^{-\frac{|\xi|^2}{2}}.$$

2.2 Application to L^p spaces

Let us devote to the Fourier transformation

$$F(f)(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Let $f \in L^1(\mathbb{R}^n)$. Then $F(f)$ belongs to $L^\infty(\mathbb{R}^n)$. Moreover, $F(f)$ is continuous on $\mathbb{R}^n$ and $\lim_{\xi \to \infty} F(f)(\xi) = 0$.

It holds the following convolution theorem: $F(f * g)(\xi) = (2\pi)^{\frac{n}{2}} F(f)(\xi) F(g)(\xi)$. Here we use the property of $L^1(\mathbb{R}^n)$ to be a Banach algebra with the operation of convolution.

Let $f \in L^2(\mathbb{R}^n)$. Then the classical definition for $F(f)$ is not applicable. Let us explain a suitable definition for $F(f)$ if f belongs to $L^2(\mathbb{R}^n)$. Let us choose $f, g \in S(\mathbb{R}^n)$. Then the classical definitions $F^{-1}(f) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x) dx$ and $F(f) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx$ are applicable. We obtain the relations

$$\int_{\mathbb{R}^n} F^{-1}(f)(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} f(\xi) \overline{F(g)(\xi)} d\xi;$$
$$\int_{\mathbb{R}^n} F(f)(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} f(\xi) \overline{F^{-1}(g)(\xi)} d\xi.$$

The first relation follows from

$$\int_{\mathbb{R}^n} F^{-1}(f)(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi\right) \overline{g(x)}dx$$
$$= \int_{\mathbb{R}^n} f(\xi)\frac{1}{(2\pi)^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{ix\xi}\overline{g(x)}dx\right) d\xi = \int_{\mathbb{R}^n} f(\xi)\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \overline{e^{-ix\xi}g(x)}dxd\xi$$
$$= \int_{\mathbb{R}^n} f(\xi)\overline{F(g)}(\xi)d\xi$$

if we use the classical definition for $F^{-1}(f)(x)$. This relation can be written as $(F^{-1}(f), g) = (f, F(g))$ for all $f, g \in S(\mathbb{R}^n)$. Let us choose now $f \in L^2(\mathbb{R}^n)$. Then the scalar product $(f, F(g))$ is defined for all $g \in S(\mathbb{R}^n)$. Taking account of the density of $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ the identity $(w, g) = (f, F(g))$ defines a functional on $L^2(\mathbb{R}^n)$. There exists a unique $w \in L^2(\mathbb{R}^n)$ such that the last relation is fulfilled for all $f \in L^2(\mathbb{R}^n)$ and $g \in S(\mathbb{R}^n)$. We define this function w as the inverse Fourier transform of $f \in L^2(\mathbb{R}^n)$. Using the second relation by a similar reasoning we are able to define the Fourier transform $F(f) \in L^2(\mathbb{R}^n)$ for a given function $f \in L^2(\mathbb{R}^n)$. Summarizing we explained $F(f)$ and $F^{-1}(f)$ for a given function $f \in L^2(\mathbb{R}^n)$ by the aid of the relations

$$(F^{-1}(f), g)_{L^2} = (f, F(g))_{L^2} \qquad \text{for all} \quad g \in S(\mathbb{R}^n),$$
$$(F(f), g)_{L^2} = (f, F^{-1}(g))_{L^2} \qquad \text{for all} \quad g \in S(\mathbb{R}^n).$$

Theorem 2.2. *The Fourier transformation is a unitary operator on $L^2(\mathbb{R}^n)$.*

Proof. By the above relations and the Fourier inversion formula for $g \in S(\mathbb{R}^n)$ it holds

$$(F^{-1}(F(f)), g)_{L^2} = (F(f), F(g))_{L^2} = (f, F^{-1}(F(g)))_{L^2} = (f, g)_{L^2}.$$

The density of $S(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ implies immediately the Fourier inversion formula for $f \in L^2(\mathbb{R}^n)$ because $F^{-1}(F(f)) = f$ from the above identity for functionals. Consequently, F maps $L^2(\mathbb{R}^n)$ onto itself. Moreover, F is isometric. Here we use the above relation with $g = F(h)$. It follows

$$(F(f), F(h))_{L^2} = (f, h)_{L^2}$$

for all $f \in L^2(\mathbb{R}^n)$ and all $h \in S(\mathbb{R}^n)$. Applying again the density argument gives the relation for all $f, h \in L^2(\mathbb{R}^n)$. □

Remark 2.2. *The formula*

$$(F(f), F(h))_{L^2} = (f, h)_{L^2} \ \textit{for } f, h \in L^2(\mathbb{R}^n)$$

is called formula of Parseval-Plancherel. We obtain $\|F(f)\|^2_{L^2} = \|f\|^2_{L^2}$ in the special case $f = h$.

An argument from interpolation theory:
We know the following properties of Fourier transformation: $f \in L^2(\mathbb{R}^n) \Rightarrow F(f) \in L^2(\mathbb{R}^n)$, and $f \in L^1(\mathbb{R}^n) \Rightarrow F(f) \in L^\infty(\mathbb{R}^n)$. By *interpolation* we conclude $f \in L^p(\mathbb{R}^n) \Rightarrow F(f) \in L^q(\mathbb{R}^n)$ for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.3. *The statement $f \in L^p(\mathbb{R}^n) \Rightarrow F(f) \in L^q(\mathbb{R}^n)$ for $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ is false!*

2.2.1 Fourier inversion formulas

Up to now we learned the Fourier inversion formula for functions belonging to $S(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. In the following we present Fourier inversion formulas for other function spaces.

• Let $f \in L^1(\mathbb{R})$ be of bounded variation on every compact interval $[a, b]$ and continuous. Then it holds

$$f(x) = \frac{1}{\sqrt{2\pi}} HW \int_{\mathbb{R}} e^{ix\xi} F(f)(\xi) d\xi.$$

Without assuming the continuity we have

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{\sqrt{2\pi}} HW \int_{\mathbb{R}} e^{ix\xi} F(f)(\xi) d\xi.$$

• Let $f \in L^p(\mathbb{R}),\ p \in [1, 2]$. Then it holds

$$f(x) = \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{ix\xi} F(f)(\xi) \chi(\varepsilon\xi) d\xi$$

with a function $\chi \in C_0^\infty(\mathbb{R}^n)$, where $\chi(\eta) = \begin{cases} 1 & |\eta| \leq 1, \\ 0 & |\eta| \geq 2. \end{cases}$

• Let $f \in L^p(\mathbb{R}),\ p \in (1, 2]$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} HW \int_{\mathbb{R}} e^{ix\xi} F(f)(\xi) d\xi.$$

2.3 Application to tempered distributions

The dual space to $S(\mathbb{R}^n)$ is denoted by $S'(\mathbb{R}^n)$. Functionals from $S'(\mathbb{R}^n)$ are called *tempered distributions*. We already applied the theory of functionals for defining the Fourier transform and inverse Fourier transform of a function from $L^2(\mathbb{R}^n)$. In $S'(\mathbb{R}^n)$ we have no scalar product. But we can use the representation of functionals $f(g)$ for $f \in S'(\mathbb{R}^n)$ and arbitrary $g \in S(\mathbb{R}^n)$ for defining the Fourier transform and inverse Fourier transform of a tempered distribution. We define

$$F(f)(g) = f(F^{-1}(g)), \quad \text{and } F^{-1}(f)(g) = f(F(g)).$$

For tempered distributions the Fourier inversion formulas

$$F^{-1}(F(f))(g) = F(f)(F(g)) = f(F^{-1}(F(g))) = f(g), \quad \text{and } F^{-1}(F(f)) = f$$

are valid. The convolution theorem holds for tempered distributions $G \in S'(\mathbb{R}^n)$, $T \in E'(\mathbb{R}^n)$: $F(G * T) = \sqrt{2\pi}^n F(G)\ F(T)$. Here we denote by $E'(\mathbb{R}^n)$ the subspace of tempered distributions having compact support.

Example 2.2. *Fourier transform of Dirac's distribution δ_0*

Dirac's distribution δ_0 has compact support $\{0\}$, so it belongs to $E'(\mathbb{R}^n)$. Actions of δ_0 to arbitrary functions from $S(\mathbb{R}^n)$ are defined by $\delta_0(f) = f(0)$ for all functions $f \in S(\mathbb{R}^n)$. Consequently, δ_0 generates a functional on $S(\mathbb{R}^n)$. To define the Fourier transform $F(\delta_0)$ we use for all functions $f \in S(\mathbb{R}^n)$ the relations

$$F(\delta_0)(f) = \delta_0(F^{-1}(f)) = F^{-1}(f)(0) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}^n} \cdot f(x)dx = \frac{1}{\sqrt{2\pi}^n}(f).$$

Hence, we obtain $F(\delta_0) = \frac{1}{\sqrt{2\pi}^n}$.

Exercise 1 Let us determine a fundamental solution to the operator $-\Delta$ in $\mathbb{R}^3$. Such a fundamental solution is a distributional solution of $-\Delta u = \delta_0$. We apply the Fourier transformation and get $|\xi|^2 F(u) = \frac{1}{\sqrt{2\pi}^3}$. This is our auxiliary problem in the phase space. It possesses the solution $F(u) = \frac{1}{|\xi|^2\sqrt{2\pi}^3}$. The main difficulty consists in determine $F^{-1}(\frac{1}{|\xi|^2\sqrt{2\pi}^3})$. Longer calculations imply $u(x) = \frac{1}{4\pi|x|}$.

2.4 Application to H^s spaces

Let us introduce the space $H^m(\mathbb{R}^n)$, $m \in \mathbb{N}$. This is the set of functions

$$H^m(\mathbb{R}^n) = \Big\{ u \in S'(\mathbb{R}^n) : \|u\|_{H^m(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |F(u)(\xi)|^2 (1+|\xi|^2)^m d\xi \Big)^{1/2} < \infty \Big\}.$$

Let us explain this definition. We know that $\partial_x^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$ and arbitrary $u \in H^m(\mathbb{R}^n)$, $m \in \mathbb{N}$. Applying the formula of Parseval-Plancherel and due to the rules for the Fourier transformation we have $\|F(\partial_x^\alpha u)\|_{L^2(\mathbb{R}^n)} = \|\xi^\alpha F(u)\|_{L^2(\mathbb{R}^n)}$ for all $|\alpha| \le m$. Such functions u satisfy the relations $\int_{\mathbb{R}^n} |F(u)(\xi)|^2 (1+|\xi|^2)^m d\xi < \infty$ and conversely. If a given function satisfies the last relation, then $\partial_x^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$. The definition of H^m by using the behavior of the Fourier transform has an advantage. It can be generalized to all real $s \in \mathbb{R}$.

Definition 2.2. *By $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, we define the set of tempered distributions*

$$H^s(\mathbb{R}^n) = \Big\{ u \in S'(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |F(u)(\xi)|^2 (1+|\xi|^2)^s d\xi \Big)^{1/2} < \infty \Big\}.$$

Applying Sobolev's imbedding theorem the space $H^s(\mathbb{R}^n)$ is imbedded in the space $C_B^2(\mathbb{R}^n)$ if $s > \frac{n}{2} + 2$. Here $C_B^2(\mathbb{R}^n)$ denotes the spaces of twice continuously differentiable functions with bounded derivatives. For $s \geq 0$ all elements from $H^s(\mathbb{R}^n)$ belong to $L^2(\mathbb{R}^n)$. For $s < 0$ we have spaces of distributions. To which space H^s does Dirac's distribution δ_0 belong to?

3 Representation of solutions to wave models - application of partial Fourier transformation

3.1 Classical wave equation

We are interested again in the Cauchy problem

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; x \in \mathbb{R}^n, \;\; n \geq 1.$$

After application of *partial Fourier transformation* $\big(v(t,\xi) = F_{x\to\xi}(u(t,x))\big)$ we get the auxiliary Cauchy problem

$$v_{tt} + |\xi|^2 v = 0, \;\; v(0,\xi) = F(\varphi)(\xi), \;\; v_t(0,\xi) = F(\psi)(\xi)$$

for an *ordinary differential equation depending on the parameter* $\xi \in \mathbb{R}^n$. For $\xi \neq 0$ we have the general solution

$$v(t,\xi) = c_1(\xi)e^{-i|\xi|t} + c_2(\xi)e^{i|\xi|t}.$$

The Cauchy conditions imply

$$c_1(\xi) + c_2(\xi) = F(\varphi)(\xi), \; -i|\xi|c_1(\xi) + i|\xi|c_2(\xi) = F(\psi)(\xi).$$

It follows

$$c_1(\xi) = \frac{1}{2}\,F(\varphi)(\xi) - \frac{1}{2i|\xi|}\,F(\psi)(\xi), \quad c_2(\xi) = \frac{1}{2}\,F(\varphi)(\xi) + \frac{1}{2i|\xi|}\,F(\psi)(\xi).$$

Setting these coefficients into the general solution gives

$$v(t,\xi) = \cos(|\xi|t)F(\varphi)(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\,F(\psi)(\xi).$$

Supposing for a moment the validity of the *Fourier inversion formula* $u(t,x) = F^{-1}_{\xi\to x}\big(F_{x\to\xi}(u(t,x))\big)$ (this relation must be checked at the end of our considerations) we arrive at the following representation for u:

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\cos(|\xi|t)F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}\,F(\psi)(\xi)\right).$$

One can also use the equivalent representation

$$\begin{aligned} u(t,x) &= F^{-1}_{\xi\to x}\Big(e^{-i|\xi|t}\frac{1}{2}\,F(\varphi)(\xi)\Big) - F^{-1}_{\xi\to x}\Big(e^{-i|\xi|t}\frac{1}{2i|\xi|}\,F(\psi)(\xi)\Big) \\ &+ F^{-1}_{\xi\to x}\Big(e^{i|\xi|t}\frac{1}{2}\,F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\Big(e^{i|\xi|t}\frac{1}{2i|\xi|}\,F(\psi)(\xi)\Big). \end{aligned}$$

This representation consists of so-called *Fourier multipliers*

$$F^{-1}_{\xi\to x}\left(e^{i\phi(t,\xi)}a(t,\xi)\;F(u_0)(\xi)\right).$$

Here $\phi = \phi(t,\xi)$ is the so-called *phase function* and $a = a(t,\xi)$ is the so-called *amplitude function*.

For given data we assume $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$ with $s \in \mathbb{R}$.

Exercise 2 Recall the notation $C\big([0,T],H^1(\mathbb{R}^n)\big)\cap C^1\big([0,T],L^2(\mathbb{R}^n)\big)$.

Theorem 3.1. *Let $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $n \geq 1$ in the Cauchy problem*

$$u_{tt} - \Delta u = 0,\;\; u(0,x) = \varphi(x),\;\; u_t(0,x) = \psi(x).$$

Then there exists a unique solution $u \in C\big([0,T],H^s(\mathbb{R}^n)\big)\cap C^1\big([0,T],H^{s-1}(\mathbb{R}^n)\big)$.

Proof. The solution is given in the form

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\cos(|\xi|t)F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\left(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\right)$$

if this solution satisfies the desired regularity. Let us transfer the assumptions for the data into the Fourier image. Then we have

$$\begin{aligned} F(\varphi)(\xi) \in L^{2,s}, &\quad \text{that is,}\quad \langle\xi\rangle^s F(\varphi)(\xi) \in L^2, \langle\xi\rangle = (1+|\xi|^2)^{1/2}, \\ F(\psi)(\xi) \in L^{2,s-1}, &\quad \text{that is,}\quad \langle\xi\rangle^{s-1}F(\psi)(\xi) \in L^2. \end{aligned}$$

We use the following estimates:

- $|\cos(|\xi|t)| \leq 1$,
- $|\sin(|\xi|t)| \leq |\xi|t \leq |\xi|T$, for $|\xi| \leq \varepsilon$ and $t \in [0,T]$,
- $|\sin(|\xi|t)| \leq 1$ for $|\xi| \geq \varepsilon$ and $t \in [0,T]$.

Thus we can conclude

$$
\begin{aligned}
&|v(t,\xi)| \leq |F(\varphi)(\xi)| + C(\varepsilon,T)\tfrac{|F(\psi)(\xi)|}{\langle\xi\rangle},\\
&\langle\xi\rangle^s|v(t,\xi)| \leq \langle\xi\rangle^s|F(\varphi)(\xi)| + C(\varepsilon,T)\langle\xi\rangle^{s-1}|F(\psi)(\xi)|.
\end{aligned}
$$

This leads to $v \in L^\infty([0,T],L^{2,s})$. In the same way we derive $\partial_t v \in L^\infty([0,T],L^{2,s-1})$. It remains to prove $v \in C([0,T],L^{2,s}) \cap C^1([0,T],L^{2,s-1})$. The property $v \in C([0,T],L^{2,s})$ follows from $\lim\limits_{t_1\to t_2} ||v(t_1,\cdot)-v(t_2,\cdot)||_{L^{2,s}} = 0$.

Using the explicit representation of solution we conclude as follows:

$$
\begin{aligned}
&\lim_{t_1\to t_2}\int\limits_{\mathbb{R}^n} |v(t_1,\xi)-v(t_2,\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad\leq \lim_{t_1\to t_2}\int\limits_{\mathbb{R}^n}\Big|\sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad+\lim_{t_1\to t_2}\int\limits_{\mathbb{R}^n}\Big|\cos\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2\frac{1}{|\xi|^2}|F(\psi)(\xi)|^2\langle\xi\rangle^{2s}d\xi.
\end{aligned}
$$

Let $K_R(0) \subset \mathbb{R}^n$ be a sufficiently large ball around the origin with radius R. We divide the integral $\int\limits_{\mathbb{R}^n}$ in two integrals $\int\limits_{K_R(0)} + \int\limits_{\mathbb{R}^n\setminus K_R(0)}$. Using the above estimates it holds

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n}\Big|\sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad=\int\limits_{K_R(0)}\Big|\sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad+\int\limits_{\mathbb{R}^n\setminus K_R(0)}\Big|\sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad\leq\int\limits_{K_R(0)}\frac{|\xi|^2(t_1-t_2)^2}{4}|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi+\int\limits_{\mathbb{R}^n\setminus K_R(0)}|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi
\end{aligned}
$$

for $|t_1-t_2| < \varepsilon(R)$. The first integral at the right-hand side is estimated by $C_R(t_1-t_2)^2\|F(\varphi)\|^2_{L^{2,s}}$. Using the continuity of the Lebesgue measure the second integral is estimated by $\widetilde{\varepsilon}(R) \to 0$. Summarizing we obtain

$$
\begin{aligned}
&\lim_{t_1\to t_2}\int\limits_{\mathbb{R}^n}\Big|\sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big)\sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2|F(\varphi)(\xi)|^2\langle\xi\rangle^{2s}d\xi\\
&\quad\leq\lim_{t_1\to t_2} C_R(t_1-t_2)^2||F(\varphi)||^2_{L^{2,s}}+\widetilde{\varepsilon}(R)=\widetilde{\varepsilon}(R).
\end{aligned}
$$

Taking account of $\widetilde{\varepsilon}(R) \to 0$ for $R \to \infty$ we conclude

$$\lim_{t_1 \to t_2} \int\limits_{\mathbb{R}^n} \Big| \sin\Big(\frac{|\xi|(t_1+t_2)}{2}\Big) \sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2 |F(\varphi)(\xi)|^2 \langle\xi\rangle^{2s} d\xi = 0.$$

Repeating this approach yields

$$\lim_{t_1 \to t_2} \int\limits_{\mathbb{R}^n} \Big| \cos\Big(\frac{|\xi|(t_1+t_2)}{2}\Big) \sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2 \frac{|F(\psi)(\xi)|^2}{|\xi|^2} \langle\xi\rangle^{2s} d\xi = 0,$$

where we now divide $\int\limits_{\mathbb{R}^n}$ into $\int\limits_{K_\varepsilon(0)} + \int\limits_{K_R(0)\backslash K_\varepsilon(0)} + \int\limits_{\mathbb{R}^n \backslash K_R(0)}$.

Summarizing we have shown $v \in C\big([0,T], L^{2,s}\big)$. The validity of the Fourier inversion formula $u = F^{-1}_{\xi\to x}\big(F_{x\to\xi}(u(t,x))\big)$ implies $u \in C\big([0,T], H^s\big)$. An analogous reasoning brings $v \in C^1\big([0,T], L^{2,s-1}\big)$, $u \in C^1\big([0,T], H^{s-1}\big)$, respectively. Here we have again to use the inversion formula $\partial_t u = F^{-1}_{\xi\to x}\big(F_{x\to\xi}(\partial_t u(t,x))\big)$. □

Exercise 3 Carry out the step of the proof

$$\lim_{t_1 \to t_2} \int\limits_{\mathbb{R}^n} \Big| \cos\Big(\frac{|\xi|(t_1+t_2)}{2}\Big) \sin\Big(\frac{|\xi|(t_1-t_2)}{2}\Big)\Big|^2 \frac{|F(\psi)(\xi)|^2}{|\xi|^2} \langle\xi\rangle^{2s} d\xi = 0.$$

The considerations of this section show that the Cauchy problem is H^s well-posed.

Corollary 3.1. *The Cauchy problem*

$$u_{tt} - \Delta u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x),\ x \in \mathbb{R}^n,\ n \geq 1$$

is H^s well-posed, that is, to given data $\varphi \in H^s(\mathbb{R}^n)$, $\psi \in H^{s-1}(\mathbb{R}^n)$ there exists a uniquely determined solution $u \in C\big([0,T], H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T], H^{s-1}(\mathbb{R}^n)\big)$. The solution depends continuously on the data, that is, to each $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that $\|\varphi_1 - \varphi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^{s-1}} < \delta$ implies $\|u_1 - u_2\|_{C([0,T],H^s)\cap C^1([0,T],H^{s-1})} < \varepsilon$.

There exist two different ways to represent solutions of wave equations. On the one hand we know the representations from Theorems 5.1 to 5.5. On the other hand we are acquainted with representations consisting of Fourier multipliers. Is it possible to transfer one representation into another one?

Exercise 4 In the $1-d$ case we have the representation

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\big(e^{i\xi t} + e^{-i\xi t}\big)\frac{1}{2}F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\Big(\big(e^{i\xi t} - e^{-i\xi t}\big)\frac{1}{2i\xi}F(\psi)(\xi)\Big).$$

How can we get from this representation the d'Alembert's representation formula from Section 5.1.1?

From the representation

$$v(t,\xi) = \cos(|\xi|t)F(\varphi)(\xi) + \frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi) = \partial_t\Big(\frac{\sin(|\xi|t)}{|\xi|}F(\varphi)(\xi)\Big) + \frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)$$

it follows

$$u(t,x) = \partial_t F^{-1}_{\xi\to x}\Big(\frac{\sin(|\xi|t)}{|\xi|}F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\Big(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\Big).$$

Therefore we only have to understand

$$F^{-1}_{\xi\to x}\Big(\frac{\sin(|\xi|t)}{|\xi|}F(\psi)(\xi)\Big).$$

Exercise 5 What is the main difficulty in the discussion of the last Fourier multiplier? Which methods do we find in the literature to overcome these difficulties?

Question: What are the advantages or disadvantages of the application of the method of Fourier transformation to study wave equations?

Answer:

advantages: Choosing data from Sobolev spaces we have *no loss of regularity* (see Theorem 3.1). The approach is independent of the spatial dimension n.

disadvantages: Special qualitative properties of solutions of the wave equation as existence of forward or backward wave front, or finite propagation speed of perturbations or domain of dependence are difficult to understand by using Fourier multipliers in the representation of solution.

3.2 Klein-Gordon equation

The Cauchy problem for the Klein-Gordon equation is

$$u_{tt} - \Delta u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

with a constant $m^2 > 0$.

How can we define the total energy of a solution?

The mass term or potential forces to include into the total energy besides the *elastic and the kinetic energy* a third component, it is the *potential energy.* Thus we define the total energy

$$E_{KG}(u)(t) := \frac{1}{2}\int_{\mathbb{R}^n}\Big(|\nabla_x u(t,\cdot)|^2 + |u_t(t,\cdot)|^2 + m^2|u(t,x)|^2\Big)dx.$$

3.2.1 Energy estimates

Repeating the proof to Theorem 6.2 one can show the following result:

Theorem 3.2. *(conservation of energy)*
Let $u \in C\big([0,T], H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T], L^2(\mathbb{R}^n)\big)$ be a Sobolev solution of

$$u_{tt} - \Delta u + m^2 u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

with data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$. Then the conservation of the total energy holds,

$$E_{KG}(u)(t) = E_{KG}(u)(0) = \frac{1}{2}\Big(\|\psi\|_{L^2}^2 + \|\nabla\varphi\|_{L^2}^2 + m^2\|\varphi\|_{L^2}^2\Big) \quad \textit{for all } \; t \geq 0.$$

Exercise 6 Prove the statement of Theorem 3.2.

Another way to show the *conservation of energy* is to use the *partial Fourier transformation.* The Fourier transform $v(t,\xi) = F_{x\to\xi}(u(t,x))(t,\xi)$ satisfies the ordinary differential equation with parameter ξ, $v_{tt} + |\xi|^2 v + m^2 v = v_{tt} + \langle\xi\rangle_m^2 v = 0$ with $\langle\xi\rangle_m^2 = |\xi|^2 + m^2$. Taking into consideration the explicit representations of solutions for $v(t,\cdot)$ and $v_t(t,\cdot)$, the assumption $(\varphi, \nabla_x\varphi, \psi) \in L^2 \times L^2 \times L^2$ and *Parseval's formula* from the theory of Fourier transformation, then we conclude as follows (the third equality should be proved in detail):

$$\begin{aligned}
E_{KG}(u)(t) &= \frac{1}{2}\big(\|\nabla_x u(t,\cdot)\|_{L^2}^2 + \|u_t(t,\cdot)\|_{L^2}^2 + m^2\|u(t,\cdot)\|_{L^2}^2\big) \\
&= \frac{1}{2}\big(\|\,|\xi|\; v(t,\cdot)\|_{L^2}^2 + \|v_t(t,\cdot)\|_{L^2}^2 + m^2\|v(t,\cdot)\|_{L^2}^2\big) \\
&= \frac{1}{2}\big(\|\langle\xi\rangle_m v(t,\cdot)\|_{L^2}^2 + \|v_t(t,\cdot)\|_{L^2}^2\big) \\
&= \frac{1}{2}\big(\|\langle\xi\rangle_m v_0(\xi)\|_{L^2}^2 + \|v_1(\xi)\|_{L^2}^2\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^n}\big(|\nabla_x\varphi(x)|^2 + |\psi(x)|^2 + m^2|\varphi(x)|^2\big)\,dx = E_{KG}(u)(0).
\end{aligned}$$

Question: Do we have a similar statement to Theorem 6.1 if we are not interested in the *total energy*?

Answer: Let $K \subset \mathbb{R}^n$ be the closure of a domain. We define the *local energy*

$$E_{KG}(u,K)(t) := \frac{1}{2}\int_K \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 + m^2|u(t,x)|^2\Big)dx.$$

The following remarkable result holds by using the same notations as in Section 6.1.

Theorem 3.3. *(domain of dependence inequality)*
Let $(t_0, x_0) \in \mathbb{R}^{n+1}$ with $t_0 > 0$. We denote by Ω the conical domain bounded by the backward

characteristic cone with apex at (t_0, x_0) and by the plane $t = 0$. Let $u \in C^2(\overline{\Omega})$ be a classical solution of the Klein-Gordon equation $u_{tt} - \Delta u + m^2 u = 0$. Then the following inequality holds:

$$E_{KG}(u, K(x_0, t_0 - t)) \le E_{KG}(u, K(x_0, t_0)) \text{ for } t \in [0, t_0].$$

The domain of dependence property helps to get a uniqueness result.

Corollary 3.2. *The Cauchy problem*

$$u_{tt} - \Delta u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

possesses at most one classical solution $u \in C^2([0,\infty) \times \mathbb{R}^n)$ if the data are supposed to be sufficiently smooth.

Exercise 7 Prove the statement of Theorem 3.3.

3.2.2 Representation of solutions by using Fourier multipliers

We will study the Cauchy problem

$$u_{tt} - \Delta u + m^2 u = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \quad n \ge 1.$$

Applying the *partial Fourier transformation* ($v(t,\xi) = F_{x\to\xi}(u(t,x))$) we obtain the auxiliary Cauchy problem

$$v_{tt} + \langle\xi\rangle_m^2 v = 0, \quad v(0,\xi) = F(\varphi)(\xi), \quad v_t(0,\xi) = F(\psi)(\xi).$$

Analogous to the approach from Section 3.1 we have

$$v(t,\xi) = \cos(\langle\xi\rangle_m t)F(\varphi)(\xi) + \frac{\sin(\langle\xi\rangle_m t)}{\langle\xi\rangle_m}\,F(\psi)(\xi).$$

Supposing for the moment the validity of *Fourier's inversion formula* $u(t,x) = F^{-1}_{\xi\to x}\big(F_{x\to\xi}(u(t,x))\big)$ (this we have to check at the end) brings

$$u(t,x) = F^{-1}_{\xi\to x}\Big(\cos(\langle\xi\rangle_m t)F(\varphi)(\xi)\Big) + F^{-1}_{\xi\to x}\Big(\frac{\sin(\langle\xi\rangle_m t)}{\langle\xi\rangle_m}\,F(\psi)(\xi)\Big).$$

This is the desired *representation of solutions*. Let us given data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$ with $s \ge 1$. From Theorem 3.2 it follows that the solution has a total energy for all $t \ge 0$. Analogous to the proof of Theorem 3.1 one can show the following statement:

Theorem 3.4. *Under the assumptions $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \ge 1$, $n \ge 1$ the Cauchy problem*

$$u_{tt} - \Delta u + m^2 u = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x)$$

has a unique energy solution $u \in C\big([0,T], H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T], H^{s-1}(\mathbb{R}^n)\big)$. The solution depends continuously on the data.

Remark 3.1. *The statements of Theorem 3.1 and Theorem 3.4 coincide. The mass term or potential has no important influence on the regularity of solutions. But mass terms have an influence on energy estimates as one can see in Theorems 3.2 and 3.3.*

3.3 Damped wave equation

Exercise 8 Refresh your knowledge about the *damped harmonic oscillator* from the course "Ordinary differential equations".

Let us devote to the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

As for the classical wave equation we introduce the total energy

$$E_W(u)(t) = \frac{1}{2}\int_{\mathbb{R}^n} \Big(|\nabla_x u(t,x)|^2 + |u_t(t,x)|^2\Big)dx.$$

3.3.1 Energy estimates

First of all we are interested in energy estimates following from differentiation of the energy $E_W(u)(t)$ with respect to t and partial integration. It holds

$$\begin{aligned} E_W'(u)(t) &= \frac{1}{2}\int_{\mathbb{R}^n} \big(2\nabla_x u \cdot \nabla_x u_t + 2u_t u_{tt}\big)dx \\ &= \int_{\mathbb{R}^n} \big(\nabla_x u \cdot \nabla_x u_t + u_t(\Delta u - u_t)\big)dx = \int_{\mathbb{R}^n} -u_t(t,x)^2 dx \leq 0. \end{aligned}$$

Thus the energy is decreasing for increasing t. This seems to be no surprise because of the damping term. It arises the question for the behavior of the energy for $t \to \infty$. Of special interest is the question if the energy $E_W(u)(t)$ tends to 0 for $t \to \infty$. Such a behavior is called *decay*. This issue is discussed in Section 4.

3.3.2 Representation of solutions by using Fourier multipliers

We will deal with the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

Step 1 Transformation of the dissipation into a mass or a potential

We introduce the function $w = w(t,x)$ as $w(t,x) := e^{\frac{1}{2}t}u(t,x)$. Then w satisfies the partial differential equation

$$w_{tt} - \Delta w - \frac{1}{4}w = 0, \;\; w(0,x) = \varphi(x), \quad w_t(0,x) = \frac{1}{2}\varphi(x) + \psi(x).$$

In opposite to the Klein-Gordon equation it appears a *negative mass term*. This negative mass needs some special considerations.

Step 2 Application of partial Fourier transformation

The application of partial Fourier transformation gives an ordinary differential equation for $v = v(t,\xi) = F_{x\to\xi}(w)(t,\xi)$:

$$v_{tt} + \Big(|\xi|^2 - \frac{1}{4}\Big)v = 0, \;\; v(0,\xi) = v_0(\xi) = F(\varphi)(\xi), \;\; v_t(0,\xi) = v_1(\xi) = \frac{1}{2}\,F(\varphi)(\xi) + F(\psi)(\xi).$$

We carry out a distinction of cases for $\{\xi \in \mathbb{R}^n : |\xi| < \frac{1}{2}\}$, the mass term $|\xi|^2 - \frac{1}{4}$ is negative, and for $\{\xi \in \mathbb{R}^n : |\xi| > \frac{1}{2}\}$, the mass term $|\xi|^2 - \frac{1}{4}$ is positive.

Case 1 $\{\xi : |\xi| > \frac{1}{2}\}$

Using $|\xi|^2 > \frac{1}{4}$ we can define a new positive variable $|\eta|$ satisfying $|\eta|^2 := |\xi|^2 - \frac{1}{4} > 0$. So we get the ordinary differential equation $v_{tt} + |\eta|^2 v = 0$. Taking account of the results from Section 3.1 we obtain immediately the following representation of solution $v(t,\xi)$:

$$v(t,\xi) \;=\; \cos\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\;t\Big)\,v_0(\xi) + \frac{\sin\Big(\sqrt{|\xi|^2 - \frac{1}{4}}\;t\Big)}{\sqrt{|\xi|^2 - \frac{1}{4}}}\,v_1(\xi).$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

The solution for the transformed differential equation is

$$\begin{aligned} v(t,\xi) \;&=\; \left(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\right) e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}t} + \left(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\right) e^{\frac{1}{2}\sqrt{1-4|\xi|^2}t} \\ &=\; v_0(\xi)\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big) + \frac{2v_1(\xi)}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big). \end{aligned}$$

If we consider the Cauchy problem

$$u_{tt} - \Delta u + u_t = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

with data $\varphi \in H^s$ and $\psi \in H^{s-1}$, then we conclude from the above representations of solutions the next result (pay attention that only the *behavior for large frequencies is important for the regularity of solutions*, the continuity with respect to t is proved as in Theorem 3.1):

Theorem 3.5. *Let the data $\varphi \in H^s(\mathbb{R}^n)$ and $\psi \in H^{s-1}(\mathbb{R}^n)$, $s \geq 1$, $n \geq 1$ be given for the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0, \; u(0,x) = \varphi(x), \; u_t(0,x) = \psi(x).$$

Then there exists a uniquely determined energy solution $u \in C\big([0,T], H^s(\mathbb{R}^n)\big) \cap C^1\big([0,T], H^{s-1}(\mathbb{R}^n)\big)$. The solution depends continuously on the data.

Remark 3.2. *The statements of Theorem 3.1 and Theorem 3.5 coincide. The dissipation term has no important influence on the regularity of solutions. Dissipation terms have an essential influence on energy estimates, they can produce a decay of the energy. This will be explained in the next section.*

Exercise 9 Let us consider the Cauchy problem for a very large damped membrane

$$u_{tt} - c^2\Delta u + ku_t = 0,\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in \mathbb{R}^2.$$

Solve this Cauchy problem by the aid of the following transformations:

$$u(t,x) = \exp(-kt/2)w(t,x), \quad v(t,x_1,x_2,x_3) = w(t,x_1,x_2)\exp(kx_3/(2c)).$$

Exercise 10 We are interested in the Cauchy problem

$$u_{tt} - u_{xx} + \varepsilon u_t = 0,\ u(0,x,\varepsilon) = \varphi(x),\ u_t(0,x,\varepsilon) = \psi(x),\ x \in \mathbb{R}^1,$$

with sufficiently smooth data φ and ψ. Let $u = u(t,x,\varepsilon)$ be the unique solution of this Cauchy problem. Show that we have for every fixed (t,x) the relation $\lim_{\varepsilon\to 0} u(t,x,\varepsilon) = w(t,x)$, where $w = w(t,x)$ solves the Cauchy problem

$$w_{tt} - w_{xx} = 0,\ \ w(0,x) = \varphi(x),\ \ w_t(0,x) = \psi(x),\ x \in \mathbb{R}^1.$$

4 Decay behavior and decay rate for classical damped waves

The *application of the partial Fourier transformation* and a very precise *WKB analysis, the abbreviation is due to the physicists Wentzel, Kramer and Broullion* (this is a precise analysis to study the Fourier multipliers appearing in the representation of solutions) allows us to estimate in a better way than it is done in Section 3.3.1 the energy of solutions to the damped wave equation. We are able to derive an *optimal decay behavior* with an optimal *decay rate.*

Theorem 4.1. *The solution to the Cauchy problem*

$$u_{tt} - \Delta u + u_t = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x)$$

with data $\varphi \in H^1$ and $\psi \in L^2$ satisfies the following estimates:

$$\begin{aligned} \|\nabla_x u(t,\cdot)\|_{L^2} &\le C(1+t)^{-\frac{1}{2}}\big(\|\psi\|_{L^2} + \|\varphi\|_{H^1}\big), \\ \|u_t(t,\cdot)\|_{L^2} &\le C(1+t)^{-1}\big(\|\psi\|_{L^2} + \|\varphi\|_{H^1}\big), \end{aligned}$$

and consequently, the energy satisfies

$$E_W(u)(t) \le C(1+t)^{-1}\big(\|\psi\|^2_{L^2} + \|\varphi\|^2_{H^1}\big).$$

Proof. Step 1 Transformation of energy into the phase space
Let $\hat{u}$ the Fourier transform of u, that is, $\hat{u}(t,\xi) = F_{x\to\xi}(u)(t,\xi)$. As in Section 3.2.1 we can transfer the energy into the phase space as follows:

$$E_W(u)(t) = \frac{1}{2}\Big(\|\nabla_x u(t,\cdot)\|^2_{L^2} + \|u_t(t,\cdot)\|^2_{L^2}\Big) = \frac{1}{2}\Big(\||\xi|\hat{u}(t,\cdot)\|^2_{L^2} + \|\hat{u}_t(t,\cdot)\|^2_{L^2}\Big).$$

By $u(t,x) = e^{-\frac{1}{2}t}w(t,x)$ and $v(t,\xi) = F_{x\to\xi}(w)(t,\xi)$ it follows $\hat{u}(t,\xi) = e^{-\frac{1}{2}t}v(t,\xi)$. For the *elastic energy* we will use

$$|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\ |\xi|v(t,\xi),$$

for the kinetic energy we will use

$$\hat{u}_t(t,\xi) = e^{-\frac{1}{2}t}\ \Big(v_t(t,\xi)\ -\frac{1}{2}\ v(t,\xi)\Big).$$

Step 2 Estimate of the elastic energy
We will distinguish several cases.
Case 1 $\{\xi:\ |\xi| > \frac{1}{2}\}$
First we notice $|\xi|\hat{u}(t,\xi) = e^{-\frac{1}{2}t}\Big(\cos\big(\sqrt{|\xi|^2-\frac{1}{4}}\ t\big)\ |\xi|v_0(\xi) + t\ \frac{\sin\big(\sqrt{|\xi|^2-\frac{1}{4}}t\big)}{\sqrt{|\xi|^2-\frac{1}{4}}t}\ |\xi|v_1(\xi)\Big)$. This helps us to estimate the elastic energy $\|\nabla_x u(t,\cdot)\|^2_{L^2}$. We have

$$\begin{aligned}\||\xi|\hat{u}(t,\xi)\|^2_{L^2\{|\xi|>\frac{1}{2}\}} &= \int_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \le 2\Big(\int_{|\xi|>\frac{1}{2}} e^{-t}|\xi|^2|v_0(\xi)|^2 d\xi \\ &+ \int_{\frac{1}{2}<|\xi|\le 1} \underbrace{\frac{\sin^2\big(\sqrt{|\xi|^2-\frac{1}{4}}t\big)}{\sqrt{|\xi|^2-\frac{1}{4}}^2\ t^2}}_{\frac{\sin^2\alpha}{\alpha^2}\le C} t^2 e^{-t}|\xi|^2|v_1(\xi)|^2 d\xi\ + \int_{|\xi|\ge 1} \underbrace{\frac{1}{|\xi|^2-\frac{1}{4}}|\xi|^2}_{\le C} e^{-t}|v_1(\xi)|^2 d\xi\Big) \\ &\le 2e^{-t}\int_{\mathbb{R}^n} |\xi|^2|v_0(\xi)|^2 d\xi\ +\ Ct^2e^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2 d\xi\ +\ Ce^{-t}\int_{\mathbb{R}^n} |v_1(\xi)|^2 d\xi.\end{aligned}$$

Summarizing *these terms decay exponentially.* It holds

$$\int_{|\xi|>\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \le Ct^2e^{-t}\int_{\mathbb{R}^n} (|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2) d\xi.$$

Case 2 $\{\xi : |\xi| < \frac{1}{2}\}$

To estimate the elastic energy we use

$$|\xi|\hat{u}(t,\xi) = |\xi| e^{-\frac{1}{2}t}\Big(\Big(\frac{v_0(\xi)}{2} - \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{-\frac{1}{2}\sqrt{1-4|\xi|^2}t}$$

$$+\Big(\frac{v_0(\xi)}{2} + \frac{v_1(\xi)}{\sqrt{1-4|\xi|^2}}\Big)e^{\frac{1}{2}\sqrt{1-4|\xi|^2 t}}\Big)$$

$$= v_0(\xi)|\xi|\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)e^{-\frac{1}{2}t} + \frac{2v_1(\xi)|\xi|}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)e^{-\frac{1}{2}t}.$$

We divide the interval $[0,\frac{1}{2})$ in two subintervals.

a) $\{\xi : |\xi| \in [\frac{1}{4},\frac{1}{2})\}$:

Here we estimate the elastic energy as follows:

$$\begin{aligned}
|\xi||\hat{u}(t,\xi)| &= \Big|\, v_0(\xi)|\xi|\underbrace{\cosh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)}_{\le \cosh(\frac{\sqrt{3}}{4}t)} e^{-\frac{1}{2}t} + \underbrace{\frac{\sinh\Big(\frac{1}{2}\sqrt{1-4|\xi|^2}t\Big)}{\frac{1}{2}\sqrt{1-4|\xi|^2}t}t}_{\le Ct\,\cosh(\frac{\sqrt{3}}{4}t)}\; v_1(\xi)|\xi|e^{-\frac{1}{2}t}\,\Big| \\
&\le \Big|\, v_0(\xi)|\xi|\underbrace{\cosh\Big(\frac{\sqrt{3}}{4}t\Big)e^{-\frac{1}{2}t}}_{\le e^{-\delta t},\ \delta>0} + C\underbrace{v_1(\xi)|\xi|}_{\le |v_1(\xi)|}\underbrace{\cosh\Big(\frac{\sqrt{3}}{4}t\Big)\, te^{-\frac{1}{2}t}}_{\le e^{-\delta t},\ \delta>0}\,\Big|,
\end{aligned}$$

and obtain

$$\int_{\frac{1}{4}\le|\xi|<\frac{1}{2}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi \le Ce^{-\delta t}\int_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2 + |v_1(\xi)|^2)d\xi.$$

b) $\{\xi : |\xi| \in [0,\frac{1}{4})\}$:

Now we use for $|\xi| < \frac{1}{2}$ the inequality $-4|\xi|^2 \le -1 + \sqrt{1-4|\xi|^2} \le -2|\xi|^2$. With this inequality we proceed as follows:

$$\begin{aligned}
\int_{|\xi|<\frac{1}{4}} |\xi|^2|\hat{u}(t,\xi)|^2 d\xi &\le \int_{|\xi|<\frac{1}{4}}(|v_1(\xi)|^2|\xi|^2 + |v_0(\xi)|^2|\xi|^2)(\underbrace{e^{-t-\sqrt{1-4|\xi|^2}t}}_{\le e^{-t}} + \underbrace{e^{-t+\sqrt{1-4|\xi|^2}t}}_{\le e^{-2|\xi|^2 t}})d\xi \\
&\le Ce^{-t}\int_{|\xi|<\frac{1}{4}}(|v_1(\xi)|^2|\xi|^2 + |v_0(\xi)|^2|\xi|^2)d\xi \\
&\quad + C\int_{|\xi|<\frac{1}{4}}(|v_1(\xi)|^2 + |v_0(\xi)|^2)|\xi|^2 e^{-2|\xi|^2 t}d\xi.
\end{aligned}$$

By using the norm inequality $\|\cdot\|_{L^2} \le \|\cdot\|_{L^\infty}\|\cdot\|_{L^2}$ we get for the second term of the right-hand side of the last inequality

$$\begin{aligned} C\int_{|\xi|<\frac{1}{4}} &(|v_1(\xi)|^2+|v_0(\xi)|^2)|\xi|^2e^{-2|\xi|^2t}d\xi \\ &\le C\sup_{|\xi|<\frac{1}{4},\,t\ge1}\frac{t|\xi|^2}{t}e^{-2|\xi|^2t}\int_{\mathbb{R}^n}(|v_1(\xi)|^2+|v_0(\xi)|^2)d\xi \\ &\le C\frac{1}{t}\underbrace{\sup_{|\xi|<\frac{1}{4},\,t\ge1} t|\xi|^2e^{-2|\xi|^2t}}_{\le C}\int_{\mathbb{R}^n}(|v_1(\xi)|^2+|v_0(\xi)|^2)d\xi. \end{aligned}$$

Summarizing we have shown for small frequencies

$$\int_{|\xi|<\frac{1}{4}}|\xi|^2|\hat{u}(t,\xi)|^2d\xi \;\le\; C(1+t)^{-1}\int_{\mathbb{R}^n}(|v_0(\xi)|^2+|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi.$$

Step 3 Estimate of the kinetic energy
Finally, we deal with the kinetic energy. We will use the identity $\|u_t(t,\xi)\|^2_{L^2}=\|\hat{u}_t(t,\xi)\|^2_{L^2}$ with $\hat{u}_t(t,\xi)=e^{-\frac{1}{2}t}\big(v_t(t,\xi)\;-\frac{1}{2}\,v(t,\xi)\big)$.
Case 1 $\{\xi : |\xi|>\frac{1}{2}\}$
We need

$$v_t(t,\xi)=-\sqrt{|\xi|^2-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)v_0(\xi)+\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)v_1(\xi).$$

By using the last equation we obtain

$$\begin{aligned} \hat{u}_t(t,\xi) \;&=\; e^{-\frac{1}{2}t}\Big(v_1(\xi)\Big(\cos(\sqrt{|\xi|^2-\frac{1}{4}}\,t)\;-\;\frac{1}{2}\frac{\sin(\sqrt{|\xi|^2-\frac{1}{4}}\,t)}{\sqrt{|\xi|^2\;-\;\frac{1}{4}}}\Big) \\ &\quad -v_0(\xi)\Big(\frac{1}{2}\cos(\sqrt{|\xi|^2-\frac{1}{4}}\,t)\;+\;\sqrt{|\xi|^2-\frac{1}{4}}\sin(\sqrt{|\xi|^2-\frac{1}{4}}\,t)\Big)\Big). \end{aligned}$$

Repeating the reasoning to estimate the elastic energy gives

$$\begin{aligned} \|u_t(t,\cdot)\|^2_{L^2\{|\xi|>\frac{1}{2}\}} &\le C\int_{|\xi|>\frac{1}{2}}e^{-t}|v_1(\xi)|^2\underbrace{\Big(\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)-\frac{1}{2}\frac{\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)}{\sqrt{|\xi|^2\;-\;\frac{1}{4}}}\Big)^2}_{\le Ct^2}d\xi \\ &+C\int_{|\xi|>\frac{1}{2}}e^{-t}|v_0(\xi)|^2\Big(\sqrt{|\xi|^2-\frac{1}{4}}\sin\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)\;+\;\underbrace{\frac{1}{2}\cos\Big(\sqrt{|\xi|^2-\frac{1}{4}}\,t\Big)}_{\le C}\Big)^2d\xi. \end{aligned}$$

The inequality $(|\xi|^2-\frac14)\sin^2\big(\sqrt{|\xi|^2-\frac14}\,t\big)\le|\xi|^2$ brings for $\{\xi:|\xi|>\frac12\}$

$$\int_{|\xi|>\frac12}|\xi|^2|\hat u(t,\xi)|^2d\xi\le Ct^2e^{-t}\int_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi.$$

Case 2: $\{\xi:|\xi|<\frac12\}$

After the determination of $v_t(t,\xi)$ we get immediately

$$\begin{aligned}\hat u_t(t,\xi) &= \frac12 e^{-\frac12 t}\Big(\sqrt{1-4|\xi|^2}\sinh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)-\cosh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)\Big)v_0(\xi)\\ &+ e^{-\frac12 t}\Big(\cosh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)-\frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)\Big)v_1(\xi).\end{aligned}$$

We divide again the interval $[0,\frac12)$.

a) $\{\xi:|\xi|\in[\frac14,\frac12)\}$:

Here we can show the exponential decay of the energy. On the one hand we use

$$\cosh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)+\sinh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)\le 2\cosh\Big(\frac{\sqrt3}{4}t\Big),$$

on the other hand we use

$$\Big|\frac{1}{\sqrt{1-4|\xi|^2}}\sinh\Big(\frac12\sqrt{1-4|\xi|^2}t\Big)\Big|\le C_\varepsilon t\ \text{ for }\ \frac12\sqrt{1-4|\xi|^2}t\le\varepsilon.$$

Both together gives

$$\|\hat u_t(t,\xi)\|^2_{L^2\{|\xi|\in[\frac14,\,\frac12)\}}\ \le\ Ce^{-\delta t}\int_{\mathbb{R}^n}(|\xi|^2|v_0(\xi)|^2+|v_1(\xi)|^2)d\xi$$

with a suitable positive δ.

b) $\{\xi:|\xi|<\frac14\}$:

In this case we obtain

$$\begin{aligned}\hat u_t(t,\xi)=&\Big(\frac{v_0(\xi)}{4}+\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}\Big)(\sqrt{1-4|\xi|^2}-1)e^{-\frac12 t+\frac12\sqrt{1-4|\xi|^2}t}\\ &-\Big(\frac{v_0(\xi)}{4}-\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}\Big)(\sqrt{1-4|\xi|^2}+1)e^{-\frac12 t-\frac12\sqrt{1-4|\xi|^2}t}.\end{aligned}$$

Hence, we can estimate as follows:

$$|\hat u_t(t,\xi)|\le\Big|\Big(\frac{v_1(\xi)}{2\sqrt{1-4|\xi|^2}}+\frac{v_0(\xi)}{4}\Big)\Big(\underbrace{\sqrt{1-4|\xi|^2}-1}_{\le-2|\xi|^2}\Big)\underbrace{e^{-\frac12 t+\frac12\sqrt{1-4|\xi|^2}t}}_{\le e^{-|\xi|^2t},\ |\xi|<\frac12}\Big|.$$

Recalling the estimates for the elastic energy a similar approach leads to

$$\begin{aligned}
\|\hat{u}_t(t,\xi)\|^2_{L^2\{|\xi|<\frac{1}{4}\}} &\leq C\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)|\xi|^4\left(e^{-t}+e^{-2|\xi|^2t}\right)d\xi\\
&\leq Ce^{-t}\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \;+\; C\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)|\xi|^4e^{-2|\xi|^2t}d\xi\\
&\leq Ce^{-t}\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi\\
&+C\frac{1}{t^2}\underbrace{\sup_{|\xi|<\frac{1}{4},\,t\geq 1} t^2|\xi|^4e^{-2|\xi|^2t}}_{\leq c}\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi\\
&\leq Ce^{-t}\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi \;+\; \frac{C}{(1+t)^2}\int_{|\xi|<\frac{1}{4}} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi\\
&\leq \frac{C}{(1+t)^2}\int_{\mathbb{R}^n} \left(|v_1(\xi)|^2+|v_0(\xi)|^2\right)d\xi.
\end{aligned}$$

Thus all statements from the theorem are proved. □

Question: Which part of the phase space does the decay behavior of the energy determine?
Answer: The decay behavior is determined by the small frequencies.
Question: Which property do the large frequencies influence?

5 Qualitative properties of wave models - 1

5.1 Classical wave equations

5.1.1 D'Alembert's representation in $\mathbb{R}^1$

We devote to the Cauchy problem

$$u_{tt}-u_{xx}=0,\;\; u(0,x)=\varphi(x),\;\; u_t(0,x)=\psi(x).$$

A change of variables $\xi = x-t$, $\eta = x+t$ (motivated by the *notion of characteristics*) leads to $-4u_{\xi\eta}=0$. The last partial differential equation has the general solution $u = u(\xi,\eta) = u_1(\xi)+u_2(\eta)$ with arbitrary functions u_1 and u_2. The backward transformation gives $u = u(t,x) = u_1(x-t)+u_2(x+t)$. The general solution u is a linear superposition of two waves, the wave $u_1(x-t)$ is a perturbation moving with the velocity 1 to the right-hand side. The wave $u_2(x+t)$ is moving with velocity 1 to the left-hand side. Both solutions (let us suppose for the moment that u_1 and u_2 are twice differentiable in the classical sense) are called *traveling wave solutions*. Using both Cauchy conditions we obtain

$$u(0,x)=\varphi(x)=u_1(x)+u_2(x),\quad u_t(0,x)=\psi(x)=-u_1'(x)+u_2'(x).$$

Integration of the second equation yields $-u_1(x) + u_2(x) = \int_{x_0}^{x} \psi(r)dr$, x_0 is an arbitrary constant. Hence,

$$u_1(x) = \frac{1}{2}\varphi(x) + \frac{1}{2}\int_{x_0}^{x} \psi(r)dr, \quad u_2(x) = \frac{1}{2}\varphi(x) - \frac{1}{2}\int_{x_0}^{x} \psi(r)dr.$$

Summarizing we derived the so-called *d'Alembert's representation of solution*

$$u(t,x) = \frac{1}{2}\big(\varphi(x-t) + \varphi(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \psi(r)dr.$$

5.1.2 What kind of properties do we conclude from d'Alembert's representation formula?

5.1.2.1 Regularity of solutions

Let us consider the Cauchy problem

$u_{tt} - u_{xx} = 0$, $u(0,x) = \varphi(x)$, $u_t(0,x) = \psi(x)$ with data $\varphi \in C^k(\mathbb{R}^1)$ and $\psi \in C^{k-1}(\mathbb{R}^1)$.

Theorem 5.1. *The Cauchy problem possesses one and only one solution $u \in C^k([0,\infty) \times \mathbb{R}^1)$. The solution depends continuously on the data, that is, if we change φ and ψ a bit with respect to the topologies of $C^k(\mathbb{R}^1)$ and $C^{k-1}(\mathbb{R}^1)$, then the solution u changes a bit with respect to the topology of $C^k([0,\infty) \times \mathbb{R}^1)$.*

Proof. The existence of a solution is given by d'Alembert's representation formula. The uniqueness follows from the fact that the general solution of $u_{tt} - u_{xx} = 0$ is given by the formula $u(t,x) = u_1(x-t) + u_2(x+t)$. The continuous dependence of the solution from the data is concluded from the representation formula. □

Exercise 11 Explain the statement about the continuous dependence of the solution from the data by formulas!

Exercise 12 Let us consider the Cauchy problem with data $\varphi = \psi = 0$ outside of the interval $[-l,l]$. Show that to each $x_0 \in \mathbb{R}^1$ there exist constants $T(x_0)$ and U with $u(x_0,t) = U$ for $t \geq T(x_0)$. Determine these constants.

5.1.2.2 Qualitative properties of solutions

From d'Alembert's representation formula we conclude remarkable properties for the solutions of wave equations which are typical of solutions of *hyperbolic partial differential equations*. The wave equation is one representative of this class.

Finite speed of propagation of perturbations

Let us devote to the Cauchy problem with data $\varphi \in C^2(\mathbb{R}^1)$ and $\psi \in C^1(\mathbb{R}^1)$. We perturb these data by the aid of data $\varphi_s \in C^2(\mathbb{R}^1)$ and $\psi_s \in C^1(\mathbb{R}^1)$ supported on the interval $[a,b]$.

We are interested in the propagation of these perturbations. For this reason we study the Cauchy problem

$$u_{tt} - u_{xx} = 0, \;\; u(0,x) = \varphi_s(x), \;\; u_t(0,x) = \psi_s(x)$$

with $\varphi_s = \psi_s = 0$ outside of $[a,b]$. As the solution we get

$$u_s(t,x) = \frac{1}{2}\big(\varphi_s(x-t) + \varphi_s(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \psi_s(r)dr.$$

Question: When do we feel the perturbations in a point $x_0 \in \mathbb{R}^1$ lying outside of $[a,b]$? For small times t we have $u(t,x_0) = 0$ in x_0.

Answer: We feel the perturbations after finite time $T = \text{dist } (x_0, [a,b])$. This property is called *finite propagation speed of perturbations or existence of a forward wave front.*

Domain of dependence

Question: Which information about the data has an influence on the solution in a given point (t_0, x_0)?

Answer: To determine the solution $u(t_0,x_0)$ in the point (t_0,x_0) we need the datum φ in the points $x_0 - t_0$ and $x_0 + t_0$ and the datum ψ on the interval $[x_0 - t_0, x_0 + t_0]$. The interval $[x_0 - t_0, x_0 + t_0]$ is called *domain of dependence* for the solution u in the point (t_0,x_0).

Huygens' principle

The *Huygens' principle* describes the existence of a *backward wave front*, that is, the property, that in a point $x_0 \in \mathbb{R}^1$ the solution vanishes after the time $T(x_0)$ if we are interested in the propagation of perturbations located in an interval $[a,b]$. In general we cannot expect the existence of a *backward wave front* having in mind that the *domain of dependence* for the solution u in the point (t_0,x_0) is the interval $[x_0 - t_0, x_0 + t_0]$. If we choose $\psi \equiv 0$, then the solution u in (t_0,x_0) is determined by the values of φ in $(x_0 - t_0)$ and $(x_0 + t_0)$. Consequently, after the time $T = \max(x_0 - a, b - x_0)$ we have $u \equiv 0$ in x_0. Summarizing the *Huygens' principle* holds under the assumption $\psi \equiv 0$. One can relax this condition to the condition $\int_a^b \psi(r)dr = 0$.

Propagation of singularities

Let us recall d'Alembert's representation of solution to the 1-d wave equation

$$u(t,x) = \frac{1}{2}\big(\varphi(x-t) + \varphi(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \psi(r)dr.$$

Let us assume that the data φ or ψ have a *jump* in $x = x_0$. Then there will be still a jump in the solution and the jump propagates along the characteristics $x - x_0 = t$ and $x - x_0 = -t$.

If there is only a jump in ψ, then we feel it in the first derivatives of the solution. This observation can be generalized to higher-dimensional cases. Thus singularities in the data propagate along the characteristics in the 1-d case or along the characteristic cone in the higher dimensional case.
If we have an obstacle, then singularities from the data for solutions of the wave equation will be reflected. There exist special situations where the study of propagation of singularities for solutions of mixed problems, and thus the property of *reflection of singularities*, can be understood from the study of the propagation of singularities for solutions to Cauchy problems.

5.1.2.3 Wave models with sources or sinks

Let us consider the wave model

$$u_{tt} - u_{xx} = F(t,x), \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x).$$

We suppose that the source F is integrable, let us say, $F \in L^1_{\text{loc}}([0,\infty) \times \mathbb{R}^1))$. Thus we are interested in non-classical solutions. For the solution u we choose the ansatz $u = v + w$, where v and w are solutions to the Cauchy problems (here we take account of the linearity of our model)

$$\begin{aligned} &v_{tt} - v_{xx} = F(t,x), \;\; v(0,x) = 0, \;\; v_t(0,x) = 0, \\ &w_{tt} - w_{xx} = 0, \;\; w(0,x) = \varphi(x), \;\; w_t(0,x) = \psi(x). \end{aligned}$$

The Cauchy problem for w is studied in Section 5.1.1, thus let us devote to the Cauchy problem for v.

Exercise 13 Derive the representation of solution

$$v(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-t')}^{x+(t-t')} F(x',t')dx'dt'.$$

Which values of F determine the solution v in the point (t_0,x_0)? Find the domain of dependence! Denoting the domain of dependence by $\Omega(t_0,x_0)$ we conclude $\Omega(t_0,x_0) = \{(t,x) \in \mathbb{R}^{n+1} : (t,x) \in [0,t_0) \times \{|x-x_0| \leq t_0 - t\}\}$.

5.1.3 Kirchhoff's representation in $\mathbb{R}^3$

As before we are interested in the Cauchy problem

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x), \;\; x \in \mathbb{R}^3.$$

To find a solution is more complicate than in the 1-d case. A simple observation tells us the following:

Lemma 5.1. *If $u_p = u_p(t,x)$ solves the Cauchy problem*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = 0, \;\; u_t(0,x) = p(x),$$

$p = p(x)$ is sufficiently smooth, then $\partial_t u_p =: v$ solves the Cauchy problem

$$v_{tt} - \Delta v = 0, \;\; v(0,x) = p(x), \;\; v_t(0,x) = 0.$$

Exercise 14 Prove the statement of this lemma!

Corollary 5.1. *A solution $u = u(t,x)$ of*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

is representable in the form $u(t,x) = u_\psi(t,x) + \partial_t u_\varphi(t,x)$, where the data φ and ψ are supposed to be smooth.

Thus it is sufficient to derive a formula for $u_p = u_p(t,x)$.

First we sketch how to guess such a formula, then we will prove that the formula really gives a solution (see Theorem 5.2).

We consider the auxiliary Cauchy problem

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = 0, \;\; u_t(0,x) = \delta_\varepsilon(x), \;\; x \in \mathbb{R}^3,$$

where $\delta_\varepsilon(x) = (4\pi\varepsilon)^{-\frac{3}{2}} \exp(-\frac{|x|^2}{4\varepsilon}), \;\; \varepsilon > 0$. It holds $\int_{\mathbb{R}^3} \delta_\varepsilon(x)dx = 1$ and $\lim\limits_{\varepsilon \to 0} \delta_\varepsilon(x) = 0$ for all $x \neq 0$.

The data depend on the polar distance, they are radially symmetric. Then we should expect that the solution is radially symmetric, too, thus it depends only on r and t.

Exercise 15 Show that every radially symmetric solution $u = u(t,r)$ of $u_{tt} - \Delta u = 0, \; x \in \mathbb{R}^3$, is representable in the following form:

$$u(t,r) = \frac{u_1(r+t)}{r} + \frac{u_2(r-t)}{r}$$

with arbitrary given twice differentiable functions u_1, u_2 (transform the Laplace operator into polar co-ordinates). Here $u_1 = u_1(r+t)$ is called *contracting wave* and $u_2 = u_2(r-t)$ is called *expanding wave.*

Using the Cauchy conditions then one integration leads to

$$u_2(r) = \int_{\mathbb{R}} -\frac{r}{2} \, (4\pi\varepsilon)^{-\frac{3}{2}} \exp\big(-\frac{r^2}{4\varepsilon}\big)dr = \varepsilon \, (4\pi\varepsilon)^{-\frac{3}{2}} \exp\big(-\frac{r^2}{4\varepsilon}\big) + C.$$

This gives the representation of solution

$$
\begin{aligned}
&u(t,x) = I_\varepsilon(r,t) - J_\varepsilon(r,t) := \\
&\frac{1}{4\pi r}\,\frac{1}{\sqrt{4\pi\varepsilon}}\,\exp\Big(-\frac{(r-t)^2}{4\varepsilon}\Big) - \frac{1}{4\pi r}\,\frac{1}{\sqrt{4\pi\varepsilon}}\,\exp\Big(-\frac{(r+t)^2}{4\varepsilon}\Big).
\end{aligned}
$$

The data $p = p(y)$ is supposed to be continuous. Thus it is nearly constant in a small cube Δy. Consequently, the solution $u = u(t,x)$ to the data $p(y)\delta_\varepsilon(|x-y|)\Delta y$ (Δy means localization near y!) is

$$
u(t,x) = p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big)\Delta y.
$$

The superposition of all localized influences leads to

$$
u(t,x) = \int_{\mathbb{R}^3} p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big)dy.
$$

The desired formula results from

$$
u(t,x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)\Big(I_\varepsilon(|x-y|,t) - J_\varepsilon(|x-y|,t)\Big)dy \;= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)I_\varepsilon(|x-y|,t)dy.
$$

Introducing spherical harmonics we get with $y = x + \rho\,\omega$, ω is a unit vector in $\mathbb{R}^3$,

$$
\begin{aligned}
u(t,x) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} p(y)\,\frac{1}{4\pi|x-y|}\,\frac{1}{\sqrt{4\pi\varepsilon}}\exp\Big(-\frac{(|x-y|-t)^2}{4\varepsilon}\Big)dy \\
&= \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_0^\infty \exp\Big(-\frac{(\rho-t)^2}{4\varepsilon}\Big)\frac{1}{\sqrt{4\pi\varepsilon}}\Big(\frac{1}{\rho}\int_{|\omega|=1} p(x+\rho\omega)\rho^2 d\sigma_\omega\Big)d\rho.
\end{aligned}
$$

Finally, setting $\rho - t = 2\sqrt{\varepsilon}z$ and changing the order of integration it follows

$$
\begin{aligned}
u(t,x) &= \frac{1}{4\pi}\int_{|\omega|=1}\lim_{\varepsilon\to 0}\int_{\frac{-t}{2\sqrt{\varepsilon}}}^\infty p(x+(t+2\sqrt{\varepsilon}z)\omega)(t+2\sqrt{\varepsilon}z)\exp(-z^2)dz\,d\sigma_\omega \\
&= \frac{t}{4\pi}\int_{|\omega|=1} p(x+t\omega)d\sigma_\omega.
\end{aligned}
$$

The element of surface of a ball with radius t is $d\sigma_y = t^2 d\sigma_\omega$. Setting $x + t\omega = y$ we arrive at the equivalent representation

$$
u(t,x) = \frac{1}{4\pi t}\int_{S_t(x)} p(y)d\sigma_y,
$$

where $S_t(x)$ is the surface of a ball of radius t and center x.

Remark 5.1. *The above considerations serve to guess a representation of solutions to the Cauchy problem* $u_{tt} - \Delta u = 0,\ u(0,x) = 0,\ u_t(0,x) = p(x),\ x \in \mathbb{R}^3$.

Theorem 5.2. *Let* $p \in C^k(\mathbb{R}^3)$ *with* $k \geq 2$. *Then the solution of the above Cauchy problem is given by the aid of Kirchhoff's formula*

$$u_p(t,x) = \frac{1}{4\pi t} \int_{S_t(x)} p(y) d\sigma_y.$$

The solution belongs to $C^k([0,\infty) \times \mathbb{R}^3)$.

Proof. We introduce $y = x + t\alpha,\ \alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α is a unit vector in the direction $y - x$. Using $d\sigma_t = t^2 d\sigma_1$ gives

$$u_p(t,x) = \frac{t}{4\pi} \int_{S_1(0)} p(x + t\alpha) d\sigma_1.$$

Thus we get $\lim\limits_{t \to 0} u_p(t,x) = 0$. Differentiating with respect to t implies together with the supposed regularity for p the relation

$$\partial_t u_p(t,x) = \frac{1}{4\pi} \int_{S_1(0)} p(x + t\alpha) d\sigma_1 + \frac{t}{4\pi} \int_{S_1(0)} \nabla p(x + t\alpha) \cdot \alpha \ d\sigma_1.$$

From this equation it follows $\lim\limits_{t \to 0} \partial_t u_p(t,x) = p(x)$. It remains to show, that u_p solves the wave equation, that is, $\Box u_p(t,x) = 0$. We use the representation

$$\partial_t u_p(t,x) = \frac{1}{t}\ u_p(t,x) + \frac{1}{4\pi t} \int_{S_t(x)} \nabla p(y) \cdot \alpha \ d\sigma_t(y).$$

Now, using the fact that α is the exterior unit normal vector to $S_t(x)$ and applying *the Divergence Theorem* we obtain

$$\partial_t u_p(t,x) = \frac{1}{t}\ u_p(t,x) + \frac{1}{4\pi t} \int_{B(x,t)} \Delta p(y) dy.$$

Here $B(x,t) \subset \mathbb{R}^3$ denotes the ball around the center x with radius t. Differentiation with respect to t yields

$$\partial_t^2 u_p(t,x) = -\frac{1}{t^2}\ u_p(t,x) + \frac{1}{t}\ \partial_t u_p(t,x) - \frac{1}{4\pi t^2} \int_{B(x,t)} \Delta p(y) dy + \frac{1}{4\pi t} \frac{\partial}{\partial t} \int_{B(x,t)} \Delta p(y) dy.$$

Setting into this equation the above relation for $\partial_t u_p$ we obtain

$$\partial_t^2 u_p(t,x) = \frac{1}{4\pi t}\ \partial_t \int_{B(x,t)} \Delta p(y) dy.$$

Taking account of

$$\partial_t \int_{B(x,t)} \Delta p(y)dy = \partial_t \int_0^t \int_{S_r(x)} \Delta p(x+r\alpha)d\sigma_r\, dr = \int_{S_t(x)} \Delta p(x+t\alpha)d\sigma_t$$

we derive

$$\partial_t^2\, u_p(t,x) = \frac{1}{4\pi t}\int_{S_t(x)} \Delta p(y)d\sigma_t = \frac{t}{4\pi}\int_{S_1(0)} \Delta p(x+\alpha t)d\sigma_1.$$

Finally, the relation

$$\Delta u_p(t,x) = \frac{t}{4\pi}\int_{S_1(0)} \Delta p(x+\alpha t)d\sigma_1$$

guarantees that $u_p = u_p(t,x)$ is a solution of our Cauchy problem. □

Corollary 5.2. *The Cauchy problem*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

with data $\varphi \in C^k(\mathbb{R}^3)$ *and* $\psi \in C^{k-1}(\mathbb{R}^3)$ *has one solution* $u \in C^{k-1}([0,\infty)\times\mathbb{R}^3)$. *This solution is representable in the form*

$$u(t,x) = \frac{1}{4\pi t}\int_{S_t(x)} \psi(y)d\sigma_y + \partial_t\Big(\frac{1}{4\pi t}\int_{S_t(x)} \varphi(y)d\sigma_y\Big).$$

<u>Question:</u> Do we see differences between the statements of Theorem 5.1 and Corollary 5.2?

<u>Answer:</u> We have no uniqueness in the formulation of Corollary 5.2. Moreover, the solution from Corollary 5.2 belongs only to the space $C^{k-1}([0,\infty)\times\mathbb{R}^3)$. Thus *we lose one order of regularity.*

Exercise 16 (Duhamel's principle) (compare with Exercise 13)
Show that the solution u of

$$u_{tt} - \Delta u = F(t,x), \;\; u(0,x) = u_t(0,x) = 0, \;\; x \in \mathbb{R}^3,$$

is given by

$$u(t,x) = \int_0^t w(x,t,\tau)d\tau,$$

where $w = w(x,t,\tau)$ solves the following Cauchy problem:

$$w_{tt} - \Delta w = 0, \;\; w(x,\tau,\tau) = 0, \;\; w_t(x,\tau,\tau) = F(\tau,x).$$

5.1.4 General dimension

5.1.4.1 Odd space dimension Let us devote to the Cauchy problem

$$u_{tt} - \Delta u = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x),\ \ x \in \mathbb{R}^{2n+1},\ \ n \geq 1.$$

Theorem 5.3. *To given data $\varphi \in C^k(\mathbb{R}^{2n+1})$ and $\psi \in C^{k-1}(\mathbb{R}^{2n+1})$ with $k \geq n+2,\ \ n \geq 1$ there exists one solution $u \in C^{k-n}([0,\infty) \times \mathbb{R}^{2n+1})$. The solution has the representation*

$$\begin{aligned} u(t,x) &= \sum_{j=0}^{n-1} \Big((j+1)a_j t^j \partial_t^j + a_j t^{j+1} \partial_t^{j+1}\Big) \frac{1}{\omega_{2n+1}} \int_{|y|=1} \varphi(x+ty) d\sigma_y \\ &+ \sum_{j=0}^{n-1} a_j t^{j+1} \partial_t^j \, \frac{1}{\omega_{2n+1}} \int_{|y|=1} \psi(x+ty) d\sigma_y, \end{aligned}$$

where $a_j = a_j(n)$ are constants with $a_{n-1} \neq 0$, and where ω_{2n+1} denotes the measure of the unit sphere in $\mathbb{R}^{2n+1}$.

<u>Question:</u> What do we conclude from this representation of solution?

<u>Answer:</u> We obtain immediately the following properties:

- The loss of regularity is n.
- The properties of finite propagation speed of perturbations, of existence of a domain of dependence and of existence of a forward and of a backward wave front are fulfilled.

<u>Question:</u> How can we prove the uniqueness of solutions? We give no answer in the moment!

Example: If $n = 1$, then $a_0 = 1$, and we conclude Kirchhoff's representation formula in 3-d case

$$u(t,x) = (1 + t\,\partial_t)\, \frac{1}{4\pi} \int_{|y|=1} \varphi(x+ty) d\sigma_y + \frac{t}{4\pi} \int_{|y|=1} \psi(x+ty) d\sigma_y.$$

5.1.4.2 Even space dimension Let us devote to the Cauchy problem

$$u_{tt} - \Delta u = 0,\ \ u(0,x) = \varphi(x),\ \ u_t(0,x) = \psi(x),\ \ x \in \mathbb{R}^{2n},\ \ n \geq 1.$$

Theorem 5.4. *To given data $\varphi \in C^k(\mathbb{R}^{2n})$ and $\psi \in C^{k-1}(\mathbb{R}^{2n})$ with $k \geq n+2,\ \ n \geq 1$ there exists a solution $u \in C^{k-n}([0,\infty) \times \mathbb{R}^{2n})$ having the representation*

$$\begin{aligned} u(t,x) &= \sum_{j=0}^{n-1} \Big((j+1)b_j t^j \partial_t^j + b_j t^{j+1} \partial_t^{j+1}\Big) \, \frac{2\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}\Gamma(n) t^{2n-1}} \\ &\quad \times \int_0^t \, \frac{r^{2n-1}}{\omega_{2n}(t^2-r^2)^{1/2}} \, \int_{|y|=1} \varphi(x+ry) d\sigma_y dr \\ &\quad + \sum_{j=0}^{n-1} b_j t^{j+1} \partial_t^j \, \frac{2\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}\Gamma(n) t^{2n-1}} \, \int_0^t \, \frac{r^{2n-1}}{\omega_{2n}(t^2-r^2)^{1/2}} \int_{|y|=1} \psi(x+ry) d\sigma_y dr, \end{aligned}$$

where $b_j = b_j(n)$ are constants with $b_{n-1} \neq 0$, and where ω_{2n} denotes the measure of the unit sphere in $\mathbb{R}^{2n}$.

Question: What do we conclude from this representation of solution?

Answer: We obtain immediately the following properties:

- The loss of regularity is n.
- The properties of finite propagation speed of perturbations, of existence of a domain of dependence and of existence of a forward wave front are fulfilled.

Example: For $n = 1$ we get Kirchhoff's representation formula in 2-d case

$$\begin{aligned} u(t,x) &= (b_0 + b_0 t\, \partial_t)\, \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)t} \int_0^t \frac{r}{\omega_2 (t^2 - r^2)^{1/2}} \int_{|y|=1} \varphi(x+ry) d\sigma_y dr \\ & \quad + b_0\, \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)} \int_0^t \frac{r}{\omega_2 (t^2 - r^2)^{1/2}} \int_{|y|=1} \psi(x+ry) d\sigma_y dr \\ &= b_0\, \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)} \Big(\partial_t \int_{K_t(x)} \frac{\varphi(y)}{\sqrt{t^2 - |y-x|^2}} dy + \int_{K_t(x)} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} dy \Big) \end{aligned}$$

after a suitable choice of the constant $b_0 \neq 0$.

Exercise 17 One can derive Kirchhoff's representation formula in 2-d case from the Kirchhoff's formula in 3-d case. Therefore one has to apply the *method of descent*. Study in the literature the method of descent!

Let us explain the method of descent. We devote to the 2-d case. The data $\varphi(x) = \varphi(x_1, x_2)$ and $\psi(x) = \psi(x_1, x_2)$ are considered as data in $\mathbb{R}^3$ which are independent of x_3. The application of Theorem 5.2 gives

$$u_p(t,x) = \frac{1}{4\pi t} \int_{S(x_1,x_2,0,t)} p(y) d\sigma_t(y) = \frac{1}{2\pi} \int_{B(x_1,x_2,t)} \frac{p(y)}{\sqrt{t^2 - |y-x|^2}} dy.$$

To derive the last relation we have to transfer the surface integral to an integral over the domain $\{y = (y_1, y_2) : |y-x|^2 \leq t^2\}$, $x = (x_1, x_2)$. Therefore we choose the parameter representation of the upper or lower half sphere in the following way:

$$\Phi_1(y_1, y_2) = y_1,\ \Phi_2(y_1, y_2) = y_2,\ y_3 = \Phi_3(y_1, y_2) = \pm (t^2 - |y-x|^2)^{1/2}.$$

For transferring the surface element we calculate the Gauß fundamentals and obtain $\sqrt{EG - F^2} = \sqrt{1 + (\partial_{y_1} y_3)^2 + (\partial_{y_2} y_3)^2}$, and finally,

$$d\sigma_t(y) = \frac{2t dy}{\sqrt{t^2 - |y-x|^2}}.$$

Following the same ideas as in Corollary 5.2 we have the following result:

Theorem 5.5. *Under the assumptions $\varphi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$ there exists a solution $u = u(t,x) \in C^2(\mathbb{R}^2 \times [0,\infty))$ having the representation*

$$u(t,x) = \frac{1}{2\pi} \int_{B(x,t)} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} dy + \frac{\partial}{\partial t}\Big(\frac{1}{2\pi} \int_{B(x,t)} \frac{\varphi(y)}{\sqrt{t^2 - |y-x|^2}} dx \Big).$$

Applying the energy method (see Section 6.1) this classical solution is the unique one in the set of classical solutions.

Applying the method of descent one can derive d'Alembert's representation formula for the 1-d case. Therefore we write

$$\frac{1}{2\pi} \int_{B(x_1,0,t)} \frac{p(y_1)}{\sqrt{t^2 - (y_1 - x_1)^2 - y_2^2}} d(y_1, y_2)$$

$$= \frac{1}{2\pi} \int_{x_1-t}^{x_1+t} p(y_1) \Big(\int_{-\sqrt{t^2-(y_1-x_1)^2}}^{\sqrt{t^2-(y_1-x_1)^2}} \frac{1}{\sqrt{t^2 - (y_1-x_1)^2 - y_2^2}} dy_2 \Big) dy_1 = \frac{1}{2} \int_{x_1-t}^{x_1+t} p(y_1) dy_1$$

by using

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - y_2^2}} dy_2 = \pi \text{ for all } a > 0.$$

6 Qualitative properties of wave models - 2

6.1 Energy method

The notion of *energy of solution of a wave equation* is a basic tool to derive uniqueness results for the wave models are discussed in Theorems 5.2 to 5.5. Let u be a given function from $C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$.

Then we denote by

$$E_W(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \Big) dx = \frac{1}{2} \|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla_x u(t,\cdot)\|_{L^2}^2$$

the *energy or total energy*, which depends only on the time variable t. Here $\frac{1}{2}\|u_t(t,\cdot)\|_{L^2}^2$ denotes the *kinetic energy* and $\frac{1}{2}\|\nabla_x u(t,\cdot)\|_{L^2}^2$ denotes the *elastic energy.*
If we are not so interested in the total energy, then we can define to a given set $K \subset \mathbb{R}^n$ (K is a closure of a domain) the *local energy*

$$E_W(u,K)(t) := \frac{1}{2} \int_K \Big(|u_t(t,x)|^2 + |\nabla_x u(t,x)|^2 \Big) dx.$$

Let $(t_0, x_0), t_0 > 0$, be a fixed point in $\mathbb{R}^{n+1}$. Then the set $\{(t,x) : |x - x_0| = |t - t_0|\}$ describes the lateral surface of a double cone with apex at (t_0, x_0). The *forward (backward)* *characteristic cone* is for $t \geq t_0$ ($t \leq t_0$) the upper (lower) cone with apex at (t_0, x_0). Let $T \leq t_0$. The part of the plane $t = T$ lying inside the backward characteristic cone will be denoted by $K(x_0, t_0 - T)$. This part is a closed ball around the center $x = x_0$ with radius $t_0 - T$. The following remarkable statement holds:

Theorem 6.1. *(domain of dependence inequality)*
Let $(t_0, x_0) \in \mathbb{R}^{n+1}$ with $t_0 > 0$. We denote by Ω the conical domain bounded by the backward characteristic cone with apex at (t_0, x_0) and by the plane $t = 0$. Let $u \in C^2(\overline{\Omega})$ be a classical solution of the wave equation $u_{tt} - \Delta u = 0$. Then the following inequality holds:

$$E_W(u, K(x_0, t_0 - t)) \leq E_W(u, K(x_0, t_0)) \text{ for } t \in [0, t_0].$$

Proof. Let Ω_T be the part of Ω below the plane $t = T$ and let C_T be the lateral surface of Ω_T. The energy method is basing on the identity

$$2u_t \Box u = -\nabla_x \cdot (2u_t \nabla_x u) + (|\nabla_x u|^2 + u_t^2)_t = 0.$$

It holds

$$0 = \int_{\Omega_T} \big(\nabla_x \cdot (2u_t \nabla_x u) - (|\nabla_x u|^2 + u_t^2)_t \big) d(x,t).$$

The integrand is equal to the divergence of the vector field $(2u_t \nabla_x u, -(|\nabla_x u|^2 + u_t^2))$. Applying the Divergence Theorem we obtain

$$0 = \int_{\partial \Omega_T} \big(2u_t \, \nabla_x u, -(|\nabla_x u|^2 + u_t^2) \big) \cdot \vec{n} \; d\sigma,$$

where $\vec{n}$ is the exterior unit normal vector to $\partial \Omega_T$. The surface $\partial \Omega_T$ consists of three parts. We study how the above integral can be written on each of the three parts.

a) Top ball $\quad K(x_0, t_0 - T) : \vec{n} = (0, \cdots, 0, 1)$. The above integral reduces to $-\int_{K(x_0, t_0 - T)} (|\nabla_x u|^2 + u_t^2) dx$.

b) Bottom ball $\quad K(x_0, t_0) : \vec{n} = (0, 0, \cdots, 0, -1)$. The above integral reduces to $\int_{K(x_0, t_0)} (|\nabla_x u|^2 + u_t^2) dx$.

c) Lateral surface C_T: The above integral reduces to
$\int_{C_T} (2u_t \; \nabla_x u, -(|\nabla_x u|^2 + u_t^2)) \cdot \vec{\eta} \; d\sigma$
$= \sqrt{2} \int_{C_T} \Big(2u_t u_{x_1} \eta_{n+1} \eta_1 + \cdots + 2u_t u_{x_n} \eta_{n+1} \eta_n - (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) \eta_{n+1}^2 \Big) d\sigma$
$= -\sqrt{2} \int_{C_T} \Big(u_{x_1} \eta_{n+1} - u_t \eta_1)^2 + \cdots + (u_{x_n} \eta_{n+1} - u_t \eta_n)^2 \Big) d\sigma \leq 0.$
Here we used $\eta_{n+1}^2 = \eta_1^2 + \cdots + \eta_n^2$.

Summarizing we have shown

$$\int_{K(x_0,t_0-T)} \Big(\big|\nabla_x u(\cdot,t)\big|^2 + u_t(\cdot,t)^2\Big)\Big|_{t=T} dx \le \int_{K(x_0,t_0)} (|\nabla_x u(\cdot,0)|^2 + u_t(\cdot,0)^2)dx.$$

Hence, the statement is proved. □

Summarizing the statements from Theorems 5.2 to 5.5 and 6.1 we conclude the next result.

Corollary 6.1. *The Cauchy problem*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x)$$

has a unique classical solution $u \in C^{k-n}([0,\infty)\times\mathbb{R}^{2n+1})$, $u \in C^{k-n}([0,\infty)\times\mathbb{R}^{2n})$, *respectively, for* $k \ge n+2$ *and* $n \ge 1$.

Exercise 19 Use Duhamel's principle and Kirchhoff's representation of solution to derive a solution to the Cauchy problem

$$u_{tt} - \Delta u = F(t,x), \;\; u(0,x) = u_t(0,x) = 0, \;\; x \in \mathbb{R}^3.$$

We assume $F \in C^2([0,T], C^2(\mathbb{R}^3))$. Why?

Exercise 20 Find the solution of the Cauchy problem

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = 1, \;\; u_t(0,x) = \frac{1}{1+|x|^2}, \;\; x \in \mathbb{R}^3.$$

Try to find two different ways to derive the representation of the solution.

Theorem 6.2. *(conservation of energy)*
Let $u \in C\big([0,T], H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T], L^2(\mathbb{R}^n)\big)$ *be a Sobolev solution of*

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi(x), \;\; u_t(0,x) = \psi(x),$$

with data $\varphi \in H^1(\mathbb{R}^n)$ *and* $\psi \in L^2(\mathbb{R}^n)$. *Then it holds*

$$E_W(u)(t) = E_W(u)(0) = \frac{1}{2}\big(\|\psi\|^2_{L^2} + \|\nabla\varphi\|^2_{L^2}\big) \quad \text{for all } \; t \ge 0.$$

Proof. Using the density of the function space $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ we are able to approximate the given data $\varphi \in H^1(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n)$ by sequences of data $\{\varphi_k\}$, $\{\psi_k\}$ with $\varphi_k, \psi_k \in C_0^\infty(\mathbb{R}^n)$. We consider the family of auxiliary problems

$$u_{tt} - \Delta u = 0, \;\; u(0,x) = \varphi_k(x), \;\; u_t(0,x) = \psi_k(x).$$

From the Theorems 5.1 to 5.5 we obtain a unique solution $u_k \in C^\infty([0,T], C_0^\infty(\mathbb{R}^n))$. Differentiating $E_W(u_k)(t)$ gives

$$E'_W(u_k)(t) = \int_{\mathbb{R}^n} \Big(\partial_t u_k(t,x)\partial_t^2 u_k(t,x) + \nabla_x u_k(t,x)\cdot\nabla_x\partial_t u_k(t,x)\Big)dx.$$

For each $t \in [0,T]$ the function $u_k(t,\cdot)$ belongs to $C_0^\infty(\mathbb{R}^n)$. After partial integration (all boundary integrals are vanishing) we obtain immediately from the wave equation

$$E'_W(u_k)(t) = \int_{\mathbb{R}^n} \big(\partial_t u_k(t,x)\Delta u_k(t,x) - \Delta u_k(t,x)\partial_t u_k(t,x)\big)dx = 0.$$

Hence, $E_W(u_k)(t) = E_W(u_k)(0) = \frac{1}{2}\Big(\|\psi_k\|_{L^2}^2 + \|\nabla\varphi_k\|_{L^2}^2\Big)$. Together with the assumption we have $\lim\limits_{k\to\infty} E_W(u_k)(0) = E_W(u)(0)$.
From the well-posedness of the Cauchy problem in Sobolev spaces (see Theorem 3.1) it follows $\lim\limits_{k\to\infty} E_W(u_k)(t) = E_W(u)(t)$. This completes the proof. □

Remark 6.1. *We proved the energy conservation for the whole space $\mathbb{R}^n$. But the energy conservation remains true for bounded domains $G \subset \mathbb{R}^n$ and classical solutions of the wave equation satisfying a homogeneous boundary condition of Dirichlet- or Neumann type. The energy conservation holds also for unbounded domains G, for example for exterior domains, if the initial data have a compact support and if classical solutions to the wave equation satisfy a homogeneous boundary condition of Dirichlet- or Neumann type. In the proof we use that the initial data influence due to the finite propagation speed the solution only in the set $\{x \in G : |x| \leq R+t\ ,\ t \geq 0\}$. Here R denotes the radius of a ball around the origin containing the support of the data.*