1 Fourier transform

Definition 1.1 (Fourier transform)**.** For $f \in L^1(\mathbb{R}^n, \mathbb{C})$ we call its Fourier transform the function defined by the following formula

$$\widehat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi\cdot x} f(x)dx. \tag{1.1}$$

We use also the notation $\mathcal{F}f(\xi) = \widehat{f}(\xi)$.

Example 1.2. We have for any $\varepsilon > 0$

$$e^{-\varepsilon\frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}} dx. \tag{1.2}$$

We set also

$$\mathcal{F}^* f(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\mathrm{i}\xi\cdot x} f(x)dx. \tag{1.3}$$

We have what follows.

Theorem 1.3. *The following facts hold.*

(1) We have $|\widehat{f}(\xi)| \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n,\mathbb{C})}$. *So in particular we have*

$$\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^n,\mathbb{C})} \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n,\mathbb{C})}. \tag{1.4}$$

(2) (Riemann– Lebesgue Lemma) We have $\lim_{\xi\to\infty} \widehat{f}(\xi) = 0$.

(3) The bounded linear operator $\mathcal{F} : L^1(\mathbb{R}^n, \mathbb{C}) \to L^\infty(\mathbb{R}^n, \mathbb{C})$ *has values in the following space* $C_0(\mathbb{R}^n, \mathbb{C}) \subset L^\infty(\mathbb{R}^n, \mathbb{C})$

$$C_0(\mathbb{R}^n, \mathbb{C}) = \{g \in C^0(\mathbb{R}^n, \mathbb{C}) : \lim_{x\to\infty} g(x) = 0\}. \tag{1.5}$$

(4) $\mathcal{F}$ *defines an isomorphism of the space of Schwartz functions* $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ *into itself.*

(5) $\mathcal{F}$ *defines an isomorphism of the space of tempered distributions* $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ *into itself.*

(6) For $f, g \in L^1(\mathbb{R}^n, \mathbb{C})$ *we have*

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{n}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

□

Theorem 1.4 (Fourier transform in L^2)**.** *The following facts hold.*

(1) *For a function $f \in C_c(\mathbb{R}^n, \mathbb{C})$ we have that $\widehat{f} \in L^2(\mathbb{R}^n, \mathbb{C})$ and $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. An operator*

$$\mathcal{F} : L^2(\mathbb{R}^n, \mathbb{C}) \to L^2(\mathbb{R}^n, \mathbb{C}) \tag{1.6}$$

remains defined. For $f \in L^2(\mathbb{R}^n, \mathbb{C})$ for any function $\varphi \in C_c(\mathbb{R}^n, \mathbb{C})$ with $\varphi = 1$ near 0 set

$$\begin{aligned} \mathcal{F}f(\xi) &:= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}\xi \cdot x} f(x) \varphi(x/\lambda) dx \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{n}{2}} \int_{|x| \le \lambda} e^{-\mathrm{i}\xi \cdot x} f(x) dx. \end{aligned} \tag{1.7}$$

Then (1.7) defines an isometric isomorphism inside $L^2(\mathbb{R}^n, \mathbb{C})$, so in particular we have

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^n, \mathbb{C})} = \|f\|_{L^2(\mathbb{R}^n, \mathbb{C})}. \tag{1.8}$$

(2) *The inverse map is defined by*

$$\begin{aligned} \mathcal{F}^* f(x) &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\mathrm{i}\xi \cdot x} f(\xi) \varphi(\xi/\lambda) d\xi \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{\frac{n}{2}} \int_{|\xi| \le \lambda} e^{\mathrm{i}\xi \cdot x} f(\xi) d\xi. \end{aligned} \tag{1.9}$$

(3) *For $f \in L^1(\mathbb{R}^n, \mathbb{C})$ the two definitions (1.1) and (1.7) of $\mathcal{F}$ coincide (by dominated convergence). Similarly, for $f \in L^1(\mathbb{R}^n, \mathbb{C})$ the two definitions (1.3) and (1.9) of $\mathcal{F}^*$ coincide.*

Theorem 1.5 (Hausdorff–Young). *For $p \in [1, 2]$ and $f \in L^p(\mathbb{R}^n, \mathbb{C})$ then (1.7) defines a function $\mathcal{F}f \in L^{p'}(\mathbb{R}^n, \mathbb{C})$ where $p' = \frac{p}{p-1}$ and an operator remains defined which satisfies*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n, \mathbb{C})} \le (2\pi)^{-n\left(\frac{1}{2} - \frac{1}{p'}\right)} \|f\|_{L^p(\mathbb{R}^n, \mathbb{C})}. \tag{1.10}$$

We know already cases $p = 2$ and $p = 1$. This implies that Theorem 1.5 is a consequence of the Marcel Riesz interpolation Theorem, which we discuss now.

Theorem 1.6 (Riesz–Thorin). *Let T be a linear map from $L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n) \cap L^{q_1}(\mathbb{R}^n)$ satisfying*

$$\|Tf\|_{L^{q_j}} \le M_j \|f\|_{L^{p_j}} \text{ for } j = 0, 1.$$

Then for $t \in (0, 1)$ and for p_t and q_t defined by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad , \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|Tf\|_{L^{q_t}} \le (M_0)^{1-t} (M_1)^t \|f\|_{L^{p_t}} \text{ for } f \in L^{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n).$$

Proof. First of all notice that if $f \in L^a \cap L^b$ with $a < b$ then $f \in L^c$ for any $c \in (a,b)$. Indeed, set $\frac{1}{c} = \frac{t}{a} + \frac{1-t}{b}$ for $t \in (0,1)$. Then $|f| = |f|^t |f|^{1-t}$ and by an extension of Hölder's inequality we have

$$\|f\|_{L^c} \le \| |f|^t \|_{L^{\frac{a}{t}}} \| |f|^{1-t} \|_{L^{\frac{b}{1-t}}} = \|f\|_{L^a}^t \|f\|_{L^b}^{1-t}.$$

Here we were alluding to the fact that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ implies

$$\|fg\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

For $p_t = p_0 = p_1 = \infty$ (in fact we can repeat a similar argument for $p_t = p_0 = p_1 = \infty$ any fixed value in $[1,\infty]$), by the above use of Hölder's inequality we have

$$\|Tf\|_{L^{q_t}} \le \|Tf\|_{L^{q_1}}^t \|Tf\|_{L^{q_0}}^{1-t} \le (M_0)^{1-t} (M_1)^t \|f\|_{L^\infty}$$

So let us suppose $p_t < \infty$. Then by density, it is not restrictive to pick f to be a simple function. It is enough to prove

$$\left| \int Tfg dx \right| \le (M_0)^{1-t} (M_1)^t \|f\|_{L^{p_t}} \|g\|_{L^{q_t'}}.$$

We already restricted to simple functions $f = \sum_{j=1}^m a_j \chi_{E_j}$ where the E_j are finite measure sets mutually disjoint. But we will assume that we can reduce to simple functions $g = \sum_{k=1}^N b_k \chi_{F_k}$ where the F_j are finite measure sets mutually disjoint. This is certainly the case if $q_t' < \infty$, by density. The case $q_t' = \infty$ reduces to the case $p_t = \infty$ by duality. In fact, see Remark 16 p. 44 [1]

$$\|T\|_{\mathcal{L}(L^{p_t}, L^{q_t})} = \|T^*\|_{\mathcal{L}(L^{q'_t}, L^{p'_t})}.$$

Notice that if both $p_0 < \infty$ and $p_1 < \infty$ and since we are treating $q_0 = q_1 = 1$ then $\|T\|_{\mathcal{L}(L^{p_j}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p'_j})} \le M_j$ and so one reduces to the case $p_t = \infty$. If, say, $p_0 = \infty$, then $\|T\|_{\mathcal{L}(L^{p_1}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, L^{p'_1})} \le M_1$ since $p_1 < \infty$, but $\|T\|_{\mathcal{L}(L^{p_0}, L^1)} = \|T^*\|_{\mathcal{L}(L^\infty, (L^\infty)')} \le M_0$, so in other words, we don't get a Lebesgue space. However, the issue is to bound for $f \in L^{p_0} \cap L^\infty$ a $T^* f \in L^1 \cap (L^\infty)' = L^1$ where $\|T^* f\|_{(L^\infty)'} = \|T^* f\|_{L^1}$, so that one can still apply the above argument used for $p_t = \infty$.
For $a_j = e^{\mathrm{i}\theta_j} |a_j|$ and $b_k = e^{\mathrm{i}\psi_k} |b_k|$ the polar representations, set

$$f_z := \sum_{j=1}^m |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{\mathrm{i}\theta_j} \chi_{E_j} \text{ with } \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$$

$$g_z := \sum_{k=1}^N |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{\mathrm{i}\psi_k} \chi_{F_k} \text{ with } \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Notice that if $q_t = 1$, then $\beta(t) = 1$ in which case g_z makes no sense. In this particular case we set $g_z = g$ instead. We consider now the function

$$F(z) = \int Tf_z g_z dx.$$

Our goal is to prove $|F(t)| \leq M_0^{1-t} M_1^t$.
$F(z)$ is holomorphic in $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \leq \operatorname{Re} z \leq 1$. Boundedness follows from estimates like

$$||a_j|^{\frac{\alpha(z)}{\alpha(t)}}| = |a_j|^{\frac{\alpha(\operatorname{Re} z)}{\alpha(t)}} \text{ which is bounded for } 0 \leq \operatorname{Re} z \leq 1.$$

We have $F(t) = \int Tfg dx$. Then (by the 3 lines lemma, see below, which yields $|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$) our Theorem is a consequence of the following two inequalities

$$|F(z)| \leq M_0 \text{ for } \operatorname{Re} z = 0 \ ;$$
$$|F(z)| \leq M_1 \text{ for } \operatorname{Re} z = 1 \ .$$

For $z = \mathrm{i}y$ we have

$$|f_{\mathrm{i}y}|^{p_0} = \sum_{j=1}^m ||a_j|^{\frac{\alpha(\mathrm{i}y)}{\alpha(t)}}|^{p_0} \chi_{E_j} = \sum_{j=1}^m ||a_j|^{\frac{\frac{1}{p_0} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}}|^{p_0} \chi_{E_j}$$
$$= \sum_{j=1}^m ||a_j|^{\mathrm{i}y p_t \left(\frac{1}{p_1} - \frac{1}{p_0}\right)} |a_j|^{\frac{p_t}{p_0}}|^{p_0} \chi_{E_j} = \sum_{j=1}^m |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}.$$

Similarly, using $1 - \beta(z) = \frac{1-z}{q_0'} + \frac{z}{q_1'}$,

$$|g_{\mathrm{i}y}|^{q_0'} = \sum_{k=1}^N ||b_k|^{\frac{1-\beta(\mathrm{i}y)}{1-\beta(t)}}|^{q_0'} \chi_{F_k} = \sum_{k=1}^N ||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'} - \frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_0'}}{\frac{1}{q_t'}}}|^{q_0'} \chi_{F_k} = \sum_{j=1}^N |b_k|^{q_t'} \chi_{F_k} = |g|^{q_t'}.$$

Then

$$|F(\mathrm{i}y)| \leq \|Tf_{\mathrm{i}y}\|_{q_0} \|g_{\mathrm{i}y}\|_{q_0'} \leq M_0 \|f_{\mathrm{i}y}\|_{p_0} \|g_{\mathrm{i}y}\|_{q_0'} = M_0 \|f\|_{p_t}^{\frac{p_t}{p_0}} \|g\|_{q_t'}^{\frac{q_t'}{q_0'}} = M_0.$$

By a similar argument

$$|f_{1+\mathrm{i}y}|^{p_1} = |f|^{p_t}$$
$$|g_{1+\mathrm{i}y}|^{q_1'} = |g|^{q_t'}.$$

Indeed by $\alpha(1 + \mathrm{i}y) = \frac{1+\mathrm{i}y}{p_1} - \frac{\mathrm{i}y}{p_0}$

$$|f_{1+\mathrm{i}y}|^{p_1} = \sum_{j=1}^m ||a_j|^{\frac{\alpha(1+\mathrm{i}y)}{\alpha(t)}}|^{p_1} \chi_{E_j} = \sum_{j=1}^m ||a_j|^{\frac{\frac{1}{p_1} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}}|^{p_1} \chi_{E_j}$$
$$= \sum_{j=1}^m ||a_j|^{\frac{p_t}{p_1}}|^{p_1} \chi_{E_j} = \sum_{j=1}^m |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}$$

and by $1-\beta(1+\mathrm{i}y) = \frac{1+\mathrm{i}y}{q_1'} - \frac{\mathrm{i}y}{q_0'}$

$$|g_{1+\mathrm{i}y}|^{q_1'} = \sum_{k=1}^{N} ||b_k|^{\frac{1-\beta(1+\mathrm{i}y)}{1-\beta(t)}}|^{q_1'} \chi_{F_k} = \sum_{k=1}^{N} ||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}} |b_k|^{\frac{\frac{1}{q_1'}}{\frac{1}{q_t'}}}|^{q_1'} \chi_{F_k} = \sum_{j=1}^{N} |b_k|^{q_t'} \chi_{F_k} = |g|^{q_t'}.$$

Finally

$$|F(1+\mathrm{i}y)| \leq \|Tf_{1+\mathrm{i}y}\|_{q_1} \|g_{1+\mathrm{i}y}\|_{q_1'} \leq M_1 \|f_{1+\mathrm{i}y}\|_{p_1} \|g_{1+\mathrm{i}y}\|_{q_1'} = M_1 \|f\|_{p_t}^{\frac{p_t}{p_1}} \|g\|_{q_t'}^{\frac{q_t'}{q_1'}} = M_1.$$

□

Here we have used the following lemma.

Lemma 1.7 (Three Lines Lemma)**.** *Let $F(z)$ be holomorphic in the strip $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \leq \operatorname{Re} z \leq 1$ and such that*

$$|F(z)| \leq M_0 \text{ for } \operatorname{Re} z = 0 \; ;$$
$$|F(z)| \leq M_1 \text{ for } \operatorname{Re} z = 1 \; .$$

Then we have $|F(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $0 < \operatorname{Re} z < 1$.

Proof. Let us start with the special case $M_0 = M_1 = 1$ and set $B := \|F\|_{L^\infty}$. Set $h_\epsilon(z) := (1+\epsilon z)^{-1}$ with $\epsilon > 0$. Since $\operatorname{Re}(1+\epsilon z) = 1 + \epsilon x \geq 1$ it follows $|h_\epsilon(z)| \leq 1$ in the strip. Furthermore $\operatorname{Im}(1+\epsilon z) = \epsilon y$ implies also $|h_\epsilon(z)| \leq |\epsilon y|^{-1}$. Consider now the two vertical lines $y = \pm B/\epsilon$ and let R be the rectangle $0 \leq x \leq 1$ and $|y| \leq B/\epsilon$. In $|y| \geq B/\epsilon$ we have

$$|F(z) h_\epsilon(z)| \leq \frac{B}{|\epsilon y|} \leq \frac{B}{|\epsilon B/\epsilon|} = 1.$$

On the other hand by the maximum modulus principle

$$\sup_R |F(z) h_\epsilon(z)| = \sup_{\partial R} |F(z) h_\epsilon(z)| \leq 1,$$

where on the horizontal sides the last inequality follows from the previous inequality and on the vertical sides follows from $|F(z)| \leq 1$ for $\operatorname{Re} z = 0, 1$ and from $|h_\epsilon(z)| \leq 1$.
Hence in the whole strip $0 \leq x \leq 1$ we have $|F(z)h_\epsilon(z)| \leq 1$ for any $\epsilon > 0$. This implies $|F(z)| \leq 1$ in the whole strip $0 \leq x \leq 1$.
In the general case $(M_0, M_1) \neq (1, 1)$ set $g(z) := M_0^{1-z} M_1^z$. Notice that

$$g(z) = e^{(1-z)\log M_0} e^{z \log M_1} \Rightarrow |g(z)| = M_0^{1-x} M_1^x \Rightarrow$$
$$\min(M_0, M_1) \leq |g(z)| \leq \max(M_0, M_1).$$

So $F(z) g^{-1}(z)$ satisfies the hypotheses of the case $M_0 = M_1 = 1$ and so $|F(z)| \leq |g(z)| = M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$

□

Another example of application of M. Riesz's Theorem is the following useful tool.

Lemma 1.8 (Young's Inequality)**.** *Let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

where

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|dy < C, \ \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x,y)|dx < C. \tag{1.11}$$

Then

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \text{ for all } p \in [1,\infty]$$

Proof. The case $p = 1,\infty$ follow immediately from (1.11). The intermediate cases from Riesz's Theorem. □

We consider now for $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and for $f \in \mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ the heat equation

$$u_t - \triangle u = 0\,, \quad u(0,x) = f(x).$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + |\xi|^2\widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{f}(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{f}(\xi)$. Notice that since $\widehat{f} \in \mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ and $e^{-t|\cdot|^2} \in \mathcal{S}(\mathbb{R}^n,\mathbb{C})$ for any $t > 0$, the last product is well defined. Furthermore, we have $\widehat{u}(t,\cdot) \in C^0([0,+\infty),\mathcal{S}'(\mathbb{R}^n,\mathbb{C}))$ and, as a consequence, since $\mathcal{F}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ also $u(t,\cdot) \in C^0([0,+\infty),\mathcal{S}'(\mathbb{R}^n,\mathbb{C}))$. We have $e^{-t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$. Then $u(t,x) = (2\pi)^{-\frac{n}{2}}G(t,\cdot) * f(x)$. In particular, for $f \in L^p(\mathbb{R}^n,\mathbb{C})$, we have

$$u(t,x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Notice that by (1.2) we have

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx = 1.$$

Theorem 1.9. *$\rho \in L^1(\mathbb{R}^n)$ be s.t. $\int \rho(x)dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-n}\rho(x/\epsilon)$. Consider $C_c(\mathbb{R}^n,\mathbb{C})$ and for each $p \in [1,\infty]$ let X_p be the closure of $C_c(\mathbb{R}^n,\mathbb{C})$ in $L^p(\mathbb{R}^n,\mathbb{C})$, so that $X_p = L^p(\mathbb{R}^n,\mathbb{C})$ for $p < \infty$ and $X_\infty = C_0(\mathbb{R}^n,\mathbb{C}) \subsetneqq L^\infty(\mathbb{R}^n,\mathbb{C})$. Then for any $f \in X_p$ we have*

$$\lim_{\epsilon\searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}^n,\mathbb{C}). \tag{1.12}$$

In particular we have

$$\lim_{t\searrow 0}(4\pi t)^{-\frac{n}{2}}e^{-\frac{|\cdot|^2}{4t}} * f = f \text{ in } L^p(\mathbb{R}^n,\mathbb{C}). \tag{1.13}$$

Proof. Clearly, (1.13) is a special case of (1.12) setting $\epsilon = \sqrt{t}$ and $\rho(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$. To prove (1.12) we start with $f \in C_c(\mathbb{R}^n, \mathbb{C})$. In this case

$$\rho_\epsilon * f(x) - f(x) = \int_{\mathbb{R}^n} (f(x - \epsilon y) - f(x))\rho(y)dy$$

so that, by Minkowski inequality and for $\Delta(y) := \|f(\cdot - y) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(x) - f(x)\|_{L^p} \le \int |\rho(y)|\Delta(\epsilon\, y)dy.$$

Now we have $\lim_{y\to 0} \Delta(y) = 0$ and $\Delta(y) \le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0} \|\rho_\epsilon * f(x) - f(x)\|_{L^p} = \lim_{\epsilon\searrow 0} \int |\rho(y)|\Delta(\epsilon\, y)dy = 0.$$

So this proves (1.12) for $f \in C_c(\mathbb{R}^n, \mathbb{C})$. The general case is proved by a density argument. □

2 Maximally dissipative operators

Sections 2–8 are taken from [2].

Definition 2.1 (Operator)**.** An (unbounded) operator on a Banach space X is a pair (A, D) with D a vector subspace of X and $A : D \to X$ a linear map. We write also $D(A) = D$ and call $D(A)$ the domain of A. The graph $G(A)$ and the range $R(A)$ of A are

$$G(A) = \{(x, Ax) \in X \times X : x \in D(A)\}$$
$$R(A) = \{Ax \in X : x \in D(A)\}.$$

Definition 2.2 (Dissipative operator)**.** An operator A in X is dissipative if $\|\lambda Ax - x\| \ge \|x\|$ for all $u \in D(A)$ and all $\lambda > 0$.

Lemma 2.3. *Let A be a dissipative operator in X, $y \in X$ and $\lambda > 0$. The there exists at most one $x \in D(A)$ s.t. $x - \lambda Ax = y$*

Proof. Indeed if $x - \lambda Ax = x' - \lambda Ax'$ then for $z = x - x'$ we have $z - \lambda Az = 0$ and the fact that A is dissipative gives $0 = \|\lambda Az - z\| \ge \|z\|$. □

Definition 2.4 (m–Dissipative operator)**.** An operator A in X is maximally dissipative (or m– dissipative from now on) if it is dissipative and if for any $y \in X$ and any $\lambda > 0$ there is $x \in X$ s.t. $x - \lambda Ax = y$.

Definition 2.5. For a given m– dissipative operator A, for any $y \in X$ and for any $\lambda > 0$ set $J_\lambda y = x$ where $x - \lambda Ax = y$. We also write $(1 - \lambda A)^{-1} = J_\lambda$.

Lemma 2.6. *$J_\lambda \in \mathcal{L}(X)$ with $\|J_\lambda\| \le 1$.*

Proof. Indeed $\|J_\lambda y\| = \|x\| \le \|\lambda Ax - x\| = \|y\|$ by Def. 2.2. $\square$

Notice that $AJ_\lambda x = J_\lambda Ax$ for all $x \in D(A)$. Indeed

$$AJ_\lambda x = \lambda^{-1}(\lambda A - 1 + 1)J_\lambda x = \lambda^{-1}(J_\lambda - 1)x = J_\lambda \lambda^{-1}(1 + \lambda A - 1)x = J_\lambda Ax.$$

Lemma 2.7. *Let an operator A in X be dissipative. The following are equivalent.*

(1) A is $m-$ dissipative.

(2) There exists a $\lambda_0 > 0$ s.t. for any $y \in X$ there is $x \in X$ s.t. $x - \lambda_0 Ax = y$.

Proof. It is enough to focus on (2) $\Rightarrow$ (1). The equation $u - \lambda Au = f$ is equivalent to

$$u - \lambda Au = f \Leftrightarrow \frac{\lambda_0}{\lambda}u - \lambda_0 Au = \frac{\lambda_0}{\lambda}f \Leftrightarrow u - \lambda_0 Au = \frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u$$

$$\Leftrightarrow u = F(u) \text{ with } F(u) := J_{\lambda_0}\left(\frac{\lambda_0}{\lambda}f + \left(1 - \frac{\lambda_0}{\lambda}\right)u\right).$$

Now we have

$$\|F(u) - F(v)\| \le k\|u - v\| \text{ for } k := \left|1 - \frac{\lambda_0}{\lambda}\right|.$$

For $\lambda \ge \lambda_0$ we have $k = \frac{\lambda_0}{\lambda} - 1 < 1$. For $\lambda < \lambda_0$

$$k = \frac{\lambda_0}{\lambda} - 1 < 1 \Leftrightarrow \frac{\lambda_0}{\lambda} < 2.$$

So we have $k \in [0, 1)$ if and only if $\lambda \in (\lambda_0/2, \infty)$. If $k \in [0, 1)$ then $u = F(u)$ has exactly one solution.
Suppose now by induction that, for $\lambda > 2^{-(n-1)}\lambda_0 =: \lambda_{n-1}$, J_λ exists. Let $\lambda' > \lambda_{n-1} > 2^{-1}\lambda'$. Then, repeating the above argument, i.e. setting $\lambda_0 = \lambda'$, it follows that $J_{\lambda_{n-1}}$ exists. But then J_λ exists for $\lambda > 2^{-1}\lambda_{n-1} = 2^{-n}\lambda_0$. So J_λ exist for any λ s.t. $\lambda > 2^{-n}\lambda_0$ for some n, that is for any $\lambda > 0$. $\square$

Example 2.8. For $\Omega \subseteq \mathbb{R}^n$ we will check later the fact that the Dirichlet Laplacian in $L^2(\Omega, \mathbb{C})$, defined by $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and with

$$D(\triangle) = \{u \in H_0^1(\Omega, \mathbb{C}) : \triangle u \in L^2(\Omega, \mathbb{C})\}$$

is $m-$ dissipative. Notice that $D(\triangle) \supseteq H_0^1(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})$. A case when equality holds is when $\partial\Omega$ is bounded and is a C^2 manifold.

Lemma 2.9. *Given A m–dissipative in X, then $\lim_{\lambda \searrow 0} \|J_\lambda x - x\| = 0 \ \forall\ x \in \overline{D(A)}$.*

Proof. Since $\|J_\lambda - 1\| \le 2$, by density we can assume $x \in D(A)$. Then

$$J_\lambda x - x = J_\lambda x - J_\lambda(1 - \lambda A)x = \lambda J_\lambda Ax.$$

So $\|J_\lambda x - x\| = \lambda\|J_\lambda Ax\| \le \lambda\|Ax\| \to 0$ for $\lambda \searrow 0$. $\square$

2.1 Some other examples of m–dissipative operators

2.1.1 $\frac{d}{dx}$ in $L^p(\mathbb{R},\mathbb{C})$

We refer to [4] p. 485. In $L^p(\mathbb{R},\mathbb{C})$ with $p\in[1,\infty]$ we consider the operator $\frac{d}{dx}$ (with $\frac{d}{dx}f$ the distributional derivative of f) with $D(\frac{d}{dx})$ the subset of $f\in L^p(\mathbb{R},\mathbb{C})$ whose distributional derivative is in $L^p(\mathbb{R},\mathbb{C})$, that is to say $W^{1,p}(\mathbb{R},\mathbb{C})$. We check that this $\frac{d}{dx}$ is m-dissipative. The case $p=2$ is easy. First of all for $\lambda>0$

$$\|f-\lambda f'\|_{L^2}=\|(1+\mathrm{i}\lambda\xi)\widehat{f}\|_{L^2}\geq\|\widehat{f}\|_{L^2}=\|f\|_{L^2}$$

so it is dissipative. Furthermore, $u=\mathcal{F}^*\frac{\widehat{f}}{1+\mathrm{i}\xi}$ solves $(1-\frac{d}{dx})u=f$, so it is m-dissipative. (Notice that also $-\frac{d}{dx}$ with $D(-\frac{d}{dx})=W^{1,2}(\mathbb{R},\mathbb{C})=H^1(\mathbb{R},\mathbb{C})$ is m-dissipative in $L^2(\mathbb{R},\mathbb{C})$.) Let us now turn to generic p. Consider the equation $u-\lambda u'=f$ or $u'-\lambda^{-1}u=-\lambda^{-1}f$. We rewrite it, at least formally, in the form $(ue^{-\lambda^{-1}x})'=-\lambda^{-1}e^{-\lambda^{-1}x}f$ and, solving formally using the "boundary condition" $\lim_{x\nearrow\infty}e^{-\lambda^{-1}x}u(x)=0$, write

$$u(x)=\lambda^{-1}\int_x^\infty e^{-\lambda^{-1}(y-x)}f(y)dy=\lambda^{-1}\int_{\mathbb{R}}e^{-\lambda^{-1}(y-x)}\chi_{\mathbb{R}_+}(y-x)f(y)dy. \tag{2.1}$$

We take this as definition of u. Then the function u belongs to $L^p(\mathbb{R})$ since

$$\|u\|_{L^p(\mathbb{R})}\leq\lambda^{-1}\|e^{-\lambda^{-1}x}\chi_{\mathbb{R}_+}\|_{L^1(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}=\|f\|_{L^p(\mathbb{R})}. \tag{2.2}$$

Finally, we claim that $u-\lambda u'=f$ is true in a distributional sense. Testing with a test function $\phi\in C_c^\infty(\mathbb{R},\mathbb{C})$

$$\begin{aligned}&\int_{\mathbb{R}}u(x)(\lambda\phi'(x)+\phi(x))dx=\int_{\mathbb{R}}\left(\lambda^{-1}\int_x^\infty e^{-\lambda^{-1}(y-x)}f(y)dy\right)(\lambda\phi'(x)+\phi(x))dx\\&=\int_{\mathbb{R}}dyf(y)\int_{-\infty}^y e^{-\lambda^{-1}(y-x)}(\phi'(x)+\lambda^{-1}\phi(x))dx=\int_{\mathbb{R}}dyf(y)\phi(y).\end{aligned} \tag{2.3}$$

Notice that the commutation in the order of integration follows because

$$\chi_{\mathbb{R}_+}(y-x)e^{-\lambda^{-1}(y-x)}f(y)(\lambda\phi'(x)+\phi(x))\in L^1(\mathbb{R}^2).$$

The last equality in (2.3) follows from the integration by parts

$$\int_{-\infty}^y e^{\lambda^{-1}(x-y)}\phi'(x)dx+\lambda^{-1}\int_{-\infty}^y e^{\lambda^{-1}(x-y)}\phi(x)dx=\phi(y).$$

(2.3) proves $u-\lambda u'=f$ in a distributional sense. Since by (2.2) we have

$$\|u\|_{L^p(\mathbb{R})}\leq\|u-\lambda u'\|_{L^p(\mathbb{R})}$$

$\frac{d}{dx}$ is dissipative if we can prove that (2.1) is the only solution of $u-\lambda u'=f$ with $u\in W^{1,p}(\mathbb{R},\mathbb{C})$. This is a consequence of the fact that the only solution $u-\lambda u'=0$ with

$u \in W^{1,p}(\mathbb{R},\mathbb{C})$ is $u=0$. It is easy see that we must have for a constant c that $u = ce^{\lambda^{-1}x}$ and for this to belong to $L^p(\mathbb{R})$ we need $c=0$. So we have shown that $\frac{d}{dx}$ is m–dissipative.

Notice that $D(\frac{d}{dx}) = W^{1,p}(\mathbb{R},\mathbb{C})$ is dense in $L^p(\mathbb{R},\mathbb{C})$ only for $p \in [1,\infty)$ and not for $p=\infty$. We will see later that this is important for the group $(e^{t\frac{d}{dx}})_{t\in\mathbb{R}}$.

For $p=\infty$ we have $W^{1,\infty}(\mathbb{R},\mathbb{C}) \subset C(\mathbb{R},\mathbb{C}) \cap L^\infty(\mathbb{R},\mathbb{C}) = \overline{C(\mathbb{R},\mathbb{C}) \cap L^\infty(\mathbb{R},\mathbb{C})} \subsetneqq L^\infty(\mathbb{R},\mathbb{C})$.

Notice that $\frac{d}{dx}$ with $D(\frac{d}{dx}) = C_0(\mathbb{R},\mathbb{C}) \cap W^{1,\infty}(\mathbb{R},\mathbb{C})$ has dense domain in $C_0(\mathbb{R},\mathbb{C})$ and is m–dissipative in $C_0(\mathbb{R},\mathbb{C})$.

Notice that also $-\frac{d}{dx}$ is m–dissipative. We know this already for $L^2(\mathbb{R},\mathbb{C})$. The case $p \neq 2$ is m-dissipative by a similar argument, redefining (2.1).

2.1.2 $\frac{d}{dx}$ in $L^p(\mathbb{R}_+,\mathbb{C})$

In $L^p(\mathbb{R}_+,\mathbb{C})$ with $p \in [1,\infty]$ we consider the operator $\frac{d}{dx}$ with $D(\frac{d}{dx}) = W^{1,p}(\mathbb{R}_+,\mathbb{C})$. We do not assume a boundary condition. We show now that $\frac{d}{dx}$ is an m-dissipative operator. We will show also that $-\frac{d}{dx}$ is not m-dissipative.

The fact that $\frac{d}{dx}$ is m-dissipative can be proved as in Subsect. 2.1.1. We define u as in (2.1) setting

$$u(x) = \chi_{[0,\infty)}(x)\lambda^{-1}\int_x^\infty e^{-\lambda^{-1}(y-x)} f(y)dy = \chi_{[0,\infty)}(x)\lambda^{-1}\int_{\mathbb{R}} e^{-\lambda^{-1}(y-x)}\chi_{\mathbb{R}_+}(y-x)f(y)dy. \tag{2.4}$$

By (2.1)–(2.2) and by (2.4) we conclude that u belongs to $L^p(\mathbb{R}_+)$ with

$$\|u\|_{L^p(\mathbb{R}_+)} \le \|f\|_{L^p(\mathbb{R})}. \tag{2.5}$$

The fact that $u - \lambda u' = f$ in a distributional sense is consequence of (2.3) when testing is done w.r.t. $\phi \in C_c^\infty(\mathbb{R}_+,\mathbb{C})$. Finally the fact that d/dx is dissipative follows from (2.5) and the fact that formula (2.4) provides the unique solution of $u - \lambda u' = f$ with $u \in W^{1,p}(\mathbb{R}_+,\mathbb{C})$. In fact any distributional solution of $u - \lambda u' = 0$ satisfies like in Sect. 2.1.1 $u = ce^{\lambda^{-1}x}$ so that $u \in L^p(\mathbb{R}_+)$ implies $c=0$.

We can ask now why $-\frac{d}{dx}$ with $D(-\frac{d}{dx}) = W^{1,p}(\mathbb{R}_+,\mathbb{C})$ is not dissipative (and so a fortiori not m–dissipative). Consider the equation $u + \lambda u' = f$. Here we can assume $f \in C_c(\mathbb{R}_+,\mathbb{C})$ so that the distributional solutions are classical solutions. Then we get $(ue^{\lambda^{-1}x})' = \lambda^{-1}e^{\lambda^{-1}x}f$ and the generic classical solution of this equation will satisfy

$$u(x) = e^{-\lambda^{-1}x}u(0) + \lambda^{-1}\int_0^x e^{-\lambda^{-1}(x-y)}f(y)dy = e^{-\lambda^{-1}x}u(0) + e^{-\lambda^{-1}x}\lambda^{-1}\int_0^x e^{\lambda^{-1}y}f(y)dy. \tag{2.6}$$

Since f has compact support, we see that for any $u(0)$ formula (2.6) yields $u \in W^{1,p}(\mathbb{R}_+,\mathbb{C})$. So for $f \in C_c(\mathbb{R}_+,\mathbb{C})$ the equation $u + \lambda u' = f$ has infinitely many solutions $u \in D(-\frac{d}{dx})$ and not just at most one as would be the case if $-\frac{d}{dx}$ was dissipative.

2.1.3 $\frac{d}{dx}$ in $L^p(\mathbb{R}_+,\mathbb{C})$ with Dirichlet condition at 0

In $L^p(\mathbb{R}_+,\mathbb{C})$ with $p\in[1,\infty]$ the operator $\frac{d}{dx}$ with Dirichlet condition at 0, that is

$$D(d/dx)=\{f\in W^{1,p}(\mathbb{R}_+,\mathbb{C}):f(0)=0\}. \tag{2.7}$$

We show now that $-\frac{d}{dx}$ is an m-dissipative operator while $\frac{d}{dx}$ is not m-dissipative. The fact that $\frac{d}{dx}$ is not m-dissipative follows readily from the fact that a solution to $u-\lambda u'=f$ has to satisfy (2.4) which, for any $f\in C_c(\mathbb{R}_+)$ nonzero and with $f\geq 0$ is s.t. $u(0)\neq 0$. We turn now to the proof that $-\frac{d}{dx}$ is m-dissipative.
Like before, consider the equation by (2.6) and using the boundary condition $u(0)=0$ write

$$u(x)=\lambda^{-1}\int_0^x e^{-\lambda^{-1}(x-y)}f(y)dy=\lambda^{-1}\int_{\mathbb{R}} e^{-\lambda^{-1}(x-y)}\chi_{\mathbb{R}_+}(x-y)\chi_{\mathbb{R}_+}(y)f(y)dy. \tag{2.8}$$

We take this as definition of u. Notice that $u\in C_0([0,\infty),\mathbb{C})$ with $u(0)=0$. Notice that (2.8) defines $u(x)$ also for $x<0$ as $u(x)=0$.

Then this function u defined in $\mathbb{R}$ belongs to $L^p(\mathbb{R})$, and so in particular its restriction on $\mathbb{R}_+$ belongs to $L^p(\mathbb{R}_+)$ since, extending in an obvious way $\chi_{\mathbb{R}_+}(x)f(x)$ on $x<0$, we get

$$\|u\|_{L^p(\mathbb{R})}\leq\lambda^{-1}\|e^{-\lambda^{-1}x}\chi_{\mathbb{R}_+}\|_{L^1(\mathbb{R})}\|\chi_{\mathbb{R}_+}f\|_{L^p(\mathbb{R})}=\|f\|_{L^p(\mathbb{R}_+)}. \tag{2.9}$$

The claim that $u+\lambda u'=f$ in a distributional sense can be proved like before testing by means of $\phi\in C_c^\infty(\mathbb{R}_+,\mathbb{C})$ and integrating by parts

$$\begin{aligned}&\int_{\mathbb{R}_+}u(x)(\phi(x)-\lambda\phi'(x))dx=\int_{\mathbb{R}_+}\lambda^{-1}\int_0^x e^{-\lambda^{-1}(x-y)}f(y)dy(\phi(x)-\lambda\phi'(x))dx\\&=\int_{\mathbb{R}_+}dyf(y)\int_y^\infty e^{-\lambda^{-1}(x-y)}(\lambda^{-1}\phi(x)-\phi'(x))dx=\int_{\mathbb{R}}dyf(y)\phi(y).\end{aligned}$$

This proves $u+\lambda u'=f$ in a distributional sense. From (2.9)

$$\|u\|_{L^p(\mathbb{R}_+)}\leq\|u+\lambda u'\|_{L^p(\mathbb{R}_+)}.$$

and so $-\frac{d}{dx}$ is dissipative. By providing a solution $u\in L^p(\mathbb{R}_+)$ of $u+\lambda u'=f$ for any $f\in L^p(\mathbb{R}_+)$ we have shown that $-\frac{d}{dx}$ is m–dissipative.

$D(-\frac{d}{dx})$ is dense in $L^p(\mathbb{R}_+,\mathbb{C})$ only for $p\in[1,\infty)$ and not for $p=\infty$. We will see later that this is important for the group $(e^{-t\frac{d}{dx}})_{t\in\mathbb{R}}$.

For $p=\infty$ $D(-\frac{d}{dx})$ is dense in

$$C_0(\mathbb{R}_+,\mathbb{C})=\{f\in C((0,\infty),\mathbb{C}):\lim_{x\nearrow\infty}f(x)=\lim_{x\searrow 0}f(x)=0\} \tag{2.10}$$

which is a closed subspace of $L^\infty(\mathbb{R}_+,\mathbb{C})$ and where it is an m–dissipative operator.

2.1.4 Laplacian

Consider the operator $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ in $L^p(\mathbb{R}^n, \mathbb{C})$ and let us set (for $\triangle f$ defined in a distributional sense)

$$D(\triangle) := \{f \in L^p(\mathbb{R}^n, \mathbb{C}) : \triangle f \in L^p(\mathbb{R}^n, \mathbb{C})\}.$$

Notice that always $W^{2,p}(\mathbb{R}^n, \mathbb{C}) \subseteq D(\triangle)$. It turns out that for $p \in (1, \infty)$ we have $W^{2,p}(\mathbb{R}^n, \mathbb{C}) = D(\triangle)$ while for $p = 1, \infty$ this is false.

Let us consider the case $p = 2$. If $u \in D(\triangle)$ then $f := (1 - \triangle)u$ is in $L^2(\mathbb{R}^n, \mathbb{C})$. Using the Fourier transform we see that

$$u := \mathcal{F}^* \left[\frac{\widehat{f}}{1 + |\xi|^2} \right] \tag{2.11}$$

The latter defines a bounded operator from $L^2(\mathbb{R}^n, \mathbb{C})$ to $H^2(\mathbb{R}^n, \mathbb{C})$. Indeed

$$\|\partial^\alpha u\|_{L^2} = \|\xi^\alpha \frac{\widehat{f}}{1 + |\xi|^2}\|_{L^2} \leq \|\widehat{f}\|_{L^2} = \|f\|_{L^2}$$

for any multi–index $|\alpha| \leq 2$. This implies that $D(\triangle) = H^2(\mathbb{R}^n, \mathbb{C})$. The operator in (2.11), which we denote by $(1 - \triangle)^{-1}$, extends into a bounded operator from $L^p(\mathbb{R}^n, \mathbb{C})$ to $W^{2,p}(\mathbb{R}^n, \mathbb{C})$ for any $p \in (1, \infty)$ and $(1 - \triangle)(1 - \triangle)^{-1} = I$. This because it can be proved, using the Calderon Zygmund theory, that

$$\|\partial^\alpha (1 - \triangle)^{-1} f\|_{L^p} = \|\mathcal{F}^*[\xi^\alpha \frac{\widehat{f}}{1 + |\xi|^2}]\|_{L^p} \leq C_p \|f\|_{L^p}$$

for $|\alpha| \leq 2$. But this is false for $p = 1, \infty$.

Having established that for $p \in (1, \infty)$ we have $D(\triangle) = W^{2,p}(\mathbb{R}^n, \mathbb{C})$ we discuss the fact that $\triangle$ is $m-$ dissipative. It is enough to show that it is dissipative.

The case $p = 2$ is easy: for $u \in H^2(\mathbb{R}^n, \mathbb{C})$ and $\lambda > 0$

$$\|u - \lambda \triangle u\|_{L^2} = \|(1 + \lambda |\xi|^2)\widehat{u}\|_{L^2} \geq \|\widehat{u}\|_{L^2} = \|u\|_{L^2}.$$

Notice that $D(\triangle) = H^2(\mathbb{R}^n, \mathbb{C})$ is dense in $L^2(\mathbb{R}^n, \mathbb{C})$. Furthermore, given $f, g \in D(\triangle)$ we have

$$\langle \triangle f, g \rangle_{L^2} = \langle |\xi|^2 \widehat{f}, \widehat{g} \rangle_{L^2} = \langle \widehat{f}, |\xi|^2 \widehat{g} \rangle_{L^2} = \langle f, \triangle g \rangle_{L^2}$$

where

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^n} f(x) \overline{g}(x) dx.$$

We will see that these facts imply that $\triangle$ is *self–adjoint* in $L^2(\mathbb{R}^n, \mathbb{C})$.
The case $p \in (1, \infty) \backslash \{2\}$ will be discussed later using the heat kernel and the Hille–Yosida-Phillips Theorem which tells us that the *generator* of a contraction semigroup is an $m-$ dissipative operator.

3 m–dissipative operators in Hilbert spaces

Definition 3.1 (Closure)**.** An operator A on a Banach space X is closed if its graph $G(A)$ is a closed subspace of $X \times X$.

Definition 3.2 (Extension)**.** Let S and T be operators on a Banach space X. S is an extension of T if $D(T) \subseteq D(S)$ and $T = S$ in $D(T)$. Equivalently, S is an extension of T if $G(T) \subseteq G(S)$.

Definition 3.3. An operator A is closable if it has a closed extension. The "smallest" closest extension is the closure of A.

Example 3.4. Consider $L^2(\mathbb{R},\mathbb{R}) \ni f(x) \overset{A}{\to} xf(x)$ where $D(A) = \{f \in L^2(\mathbb{R},\mathbb{R}) : \ xf \in L^2(\mathbb{R},\mathbb{R})\}$. Then A is closed. Indeed, let $(f_n, Af_n) \overset{n\to\infty}{\to} (f,g)$ in $L^2(\mathbb{R},\mathbb{R}) \times L^2(\mathbb{R},\mathbb{R})$. We know that this implies that there exists $X \subset \mathbb{R}$ with $\mathbb{R}\backslash X$ of 0 measure s.t. for all $x \in X$ we have

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ and } \lim_{n\to\infty} xf_n(x) = g(x).$$

Obviously this implies that $g(x) = xf(x)$ for all $x \in X$. So $f \in D(A)$. Hence A is closed.

Example 3.5. Consider $L^2(\mathbb{R},\mathbb{R}) \ni f(x) \overset{A}{\to} e^{-x^2} f(0)$ where $D(A) = C_c(\mathbb{R},\mathbb{R})$. Then notice that for any $f \in L^2(\mathbb{R},\mathbb{R})$ and any $z \in \mathbb{R}$ there exists a sequence $f_{n,z} \in C_c(\mathbb{R},\mathbb{R})$ s.t. $f_{n,z} \overset{n\to\infty}{\to} f$ in $L^2(\mathbb{R},\mathbb{R})$ and $f_{n,z}(0) = z$ for all n. This means that A is not closable, because if B was such a closure, then for $(f_{n,z}, Bf_{n,z}) = (f_{n,z}, Af_{n,z}) \to (f, ze^{-x^2})$, we would have $Bf = ze^{-x^2}$ for any $z \in \mathbb{R}$. Absurd.

Definition 3.6 (Adjoint Operator)**.** Let A be an operator with $\overline{D(A)} = X$ on a Hilbert space X (which we can always assume on $\mathbb{R}$) with inner product $\langle\, , \rangle$. Set

$$D(A^*) = \{x \in X : \exists\, y \in X \text{ s.t. } \langle Av, x\rangle = \langle v, y\rangle\, \forall\, v \in D(A)\}. \tag{3.1}$$

(Notice that for $x \in D(A^*)$ the corresponding y is unique). Then the adjoint A^* of A is defined by

$$A^* : D(A^*) \to X \text{ with } A^*x = y. \tag{3.2}$$

A is *symmetric* if A^* is an extension of A. This can be equivalently stated by $G(A) \subseteq G(A^*)$.
A is *self–adjoint* if $A^* = A$. This can be equivalently stated by $G(A) = G(A^*)$.
A is *skew–adjoint* if $A^* = -A$.

The graph $G(A^*)$ is always closed. Indeed, if $(x_n, A^*x_n) \to (\widehat{x}, \widehat{y})$ we have

$$\langle Av, \widehat{x}\rangle = \lim_n \langle Av, x_n\rangle = \lim_n \langle v, A^*x_n\rangle = \langle v, \widehat{y}\rangle\, \forall\, v \in D(A)$$

and so $\widehat{x} \in D(A^*)$ with $\widehat{y} = A^*\widehat{x}$. On the other hand, the following example shows that $D(A^*)$ not necessarily dense in X.

Example 3.7. Let $f \in L^\infty(\mathbb{R},\mathbb{R})$ with $f \notin L^2(\mathbb{R},\mathbb{R})$ and $\psi_0 \in L^2(\mathbb{R},\mathbb{R})$. Set

$$\begin{aligned} D(A) &= \{\psi \in L^2(\mathbb{R},\mathbb{R}) : \psi f \in L^1(\mathbb{R})\} \\ A\psi &= \langle f, \psi\rangle \psi_0 \end{aligned} \tag{3.3}$$

Notice that $D(A) \supseteq C_c^0(\mathbb{R},\mathbb{R})$ is dense in $L^2(\mathbb{R},\mathbb{R})$.
Suppose $\phi \in D(A^*)$. Then for all $\psi \in D(A)$

$$\langle \psi, A^*\phi\rangle = \langle A\psi, \phi\rangle = \langle\langle f,\psi\rangle\psi_0, \phi\rangle = \langle f,\psi\rangle\langle\psi_0,\phi\rangle = \langle\langle\psi_0,\phi\rangle f, \psi\rangle.$$

So $A^*\phi = \langle\psi_0,\phi\rangle f$ and this has to belong to $L^2(\mathbb{R},\mathbb{R})$. Since $f \notin L^2(\mathbb{R},\mathbb{R})$ this can happen only if $\langle\psi_0,\phi\rangle = 0$. In fact $D(A^*) = \{\psi_0\}^\perp$ where $A^*\phi = \langle\psi_0,\phi\rangle f = 0$.

Example 3.8. The operator $Af(x) = xf(x)$ in Example 3.4 is self–adjoint. First of all it is symmetric. Indeed if $g \in D(A)$ then

$$\langle g, Af\rangle = \langle g, xf\rangle = \langle xg, f\rangle \text{ for all } f \in D(A).$$

Then $g \in D(A)$ implies $g \in D(A^*)$ with $A^*g = Ag$. So A^* is an extension of A. On the other hand, let $g \in D(A^*)$. Then there exists $h \in L^2(\mathbb{R},\mathbb{R})$ s.t.

$$\langle g, Af\rangle = \langle g, xf\rangle = \langle h, f\rangle \text{ for all } f \in D(A).$$

Testing with respect of $f \in C_c^\infty(\mathbb{R},\mathbb{R}) \subset D(A)$ implies that $xg(x) = h(x)$ a.e. and so $g \in D(A)$.
In a similar fashion, given any $\varphi \in L^\infty_{loc}(\mathbb{R}^n,\mathbb{R})$, the operator $Af(x) = \varphi(x)f(x)$ where $D(A) = \{f \in L^2(\mathbb{R}^n,\mathbb{C}) : \ \varphi(\cdot)f(\cdot) \in L^2(\mathbb{R}^n,\mathbb{C})\}$ is self–adjoint.

Example 3.9. Using the last sentence we conclude that the operator $\triangle$ in $L^2(\mathbb{R}^n,\mathbb{C})$ with $D(\triangle) = H^2(\mathbb{R}^n,\mathbb{C})$ is self–adjoint in $L^2(\mathbb{R}^n,\mathbb{C})$.

Lemma 3.10. *Consider a Hilbert space X. An operator A is dissipative in X if and only if $\langle Au,u\rangle \le 0$ for all $u \in D(A)$.*

Proof. If $\langle Au,u\rangle \le 0$ for all $u \in D(A)$ then for any $\lambda > 0$ and $u \in D(A)$

$$\|u - \lambda Au\|^2 = \|u\|^2 + \lambda^2\|Au\|^2 - 2\lambda\langle Au,u\rangle \ge \|u\|^2 + \lambda^2\|Au\|^2 \ge \|u\|^2.$$

Viceversa, if A is dissipative for any $\lambda > 0$ and $u \in D(A)$

$$-2\langle Au,u\rangle + \lambda\|Au\|^2 = \lambda^{-1}\left(\|u-\lambda Au\|^2 - \|u\|^2\right) \ge 0$$

So $-\langle Au,u\rangle + 2^{-1}\lambda\|Au\|^2 \ge 0$ for any $\lambda > 0$ and $u \in D(A)$, which implies $\langle Au,u\rangle \le 0$ for any $u \in D(A)$.

□

Theorem 3.11. *Let A be a dissipative linear operator with dense domain in a Hilbert space X. Then A is m–dissipative if and only if A^* is dissipative and $G(A)$ is closed.*

Corollary 3.12. *Let A be a densely defined operator in X s.t. $G(A) \subseteq G(A^*)$ (that is, A is symmetric) and with $\langle Au, u\rangle \leq 0$ for all $u \in D(A)$. Then A is m–dissipative if and only if it is self–adjoint.*

Proof. First of all by Lemma 3.10 we know that A is dissipative.
We assume that A is self–adjoint. By $A^* = A$, A^* is dissipative. Since $G(A^*)$ is always closed, then $G(A)$ is closed. So A is a densely defined operator which is dissipative, with $G(A)$ closed and A^* dissipative. By Theorem 3.11 we conclude that A is m–dissipative
We assume now that A is m–dissipative. By Theorem 3.11 we have that A^* is dissipative. Let $(u, A^*u) \in G(A^*)$ and set $g = u - A^*u$. Since A is m–dissipative, there exists $v \in D(A)$ s.t. $g = v - Av$. Since $G(A) \subseteq G(A^*)$ we have $(v, Av) \in G(A^*)$ and $(u - v) - A^*(u - v) = 0$. But since A^* is dissipative we have $u = v$. So $G(A) = G(A^*)$ and so $A = A^*$. □

Example 3.13. For $\Omega \subseteq \mathbb{R}^n$ the Dirichlet Laplacian in $L^2(\Omega, \mathbb{C})$ is defined by $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and

$$D(\triangle) = \{u \in H^1_0(\Omega, \mathbb{C}) : \triangle u \in L^2(\Omega, \mathbb{C})\}.$$

We show that the Dirichlet Laplacian is self–adjoint and m–dissipative.
First of all it is dissipative. For $u \in D(\triangle)$ and any $\varphi \in H^1_0(\Omega)$ we claim that we have

$$\int_\Omega \varphi(x)\triangle u(x)dx = -\int_\Omega \nabla\varphi(x) \cdot \nabla u(x)dx. \tag{3.4}$$

(3.4) is true for $\varphi \in C^\infty_c(\Omega)$ and extends to all $\varphi \in H^1_0(\Omega)$ by the density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$ and by the continuity of both sides of (3.4) with respect to $\varphi \in H^1_0(\Omega)$.
(3.4) implies that $\langle u, \triangle u\rangle \leq 0 \,\forall\, u \in D(\triangle)$. This in turn is equivalent to $\triangle$ being dissipative by Lemma 3.10.
Next we show that $\triangle$ is symmetric. Indeed, $\langle u, \triangle v\rangle = \langle \triangle u, v\rangle$ for all $u, v \in D(\triangle)$ from (3.4).
Now we show that $\triangle$ is m–dissipative. Let $f \in L^2(\Omega)$. Since $H^1_0(\Omega)$ is a Hilbert space with inner product $\langle a, b\rangle_{H^1_0(\Omega)} = \langle a, b\rangle_{L^2(\Omega)} + \sum_j \langle \partial_j a, \partial_j b\rangle_{L^2(\Omega)}$, by the Frigyes Riesz representation Theorem $\exists\, u \in H^1_0(\Omega)$ s.t.

$$\langle f, \varphi\rangle_{L^2(\Omega)} = \langle u, \varphi\rangle_{H^1_0(\Omega)} \text{ for all } \varphi \in H^1_0(\Omega).$$

Restricting to $\varphi \in C^\infty_c(\Omega)$ and by the definition of $\partial^2_j u$ in the sense of distributions we obtain

$$\langle f, \varphi\rangle_{\mathcal{D}'(\Omega), C^\infty_c(\Omega)} = \langle f, \varphi\rangle_{L^2(\Omega)} = \int_\Omega u\varphi dx + \sum_{j=1}^n \int_\Omega \partial_j u \partial_j \varphi dx = \langle u - \triangle u, \varphi\rangle_{\mathcal{D}'(\Omega), C^\infty_c(\Omega)}$$

where at the extremes we have the pairing between distributions and test functions

$$\langle\ ,\ \rangle_{\mathcal{D}'(\Omega), C^\infty_c(\Omega)} : \mathcal{D}'(\Omega) \times C^\infty_c(\Omega) \to \mathbb{R}.$$

But then in $\mathcal{D}'(\Omega)$ we have $u - \triangle u = f$. This implies that $\triangle u \in L^2(\Omega)$ and since by definition we have $u \in H^1_0(\Omega)$ we conclude that $u \in D(\triangle)$. So $\triangle$ is m–dissipative. By Corollary 3.14 we conclude that $\triangle$ is also self–adjoint.

Corollary 3.14. *Let A be skew–adjoint. Then $\pm A$ are m–dissipative.*

Proof. We have $\langle Au, u\rangle = \langle A^*u, u\rangle = -\langle Au, u\rangle$ and so $\langle Au, u\rangle = 0$ for all $u \in D(A)$. So $\pm A$ is dissipative by Lemma 3.10. Since $A^* = -A$ then also A^* is dissipative. We know that $G(A^*)$ is closed, so $G(-A) = G(A^*)$ is closed and we conclude that $-A$ is m–dissipative by Theorem 3.11. Finally, the map $(x, y) \to (x, -y)$ is an isomorphism inside $X \times X$ which sends $G(A)$ in $G(-A)$. This means that $G(A)$ is closed and so A is m–dissipative by Theorem 3.11.

□

Proof of Theorem 3.11. Let A be m–dissipative with dense domain. We first prove that $G(A)$ is closed.
First of all, $G(A)$ is closed iff $G(1 - A)$ is closed. This follows from the fact that $(u, v) \to (u, u - v)$ is an isomorphism in $X \times X$, which preserves closed subspaces of $X \times X$ and which maps $G(A)$ in $G(1 - A)$.
$G(1 - A)$ is closed iff $G((1 - A)^{-1})$ is closed since they are sent one on the other by the isomorphism $(u, v) \to (v, u)$ in $X \times X$. Finally, $G((1 - A)^{-1}) = G(J_1)$ is closed because $J_1 \in \mathcal{L}(X, X)$. This completes the proof that $G(A)$ is closed.
We now show that A^* is dissipative. For $v \in D(A^*)$ we have

$$\langle A^*v, \overbrace{J_\lambda v}^{\in D(A)} \rangle = \langle v, A(1 - \lambda A)^{-1}v\rangle =$$
$$\lambda^{-1}\langle v, (\lambda A - 1 + 1)(1 - \lambda A)^{-1}v\rangle = \lambda^{-1}\langle v, \left(\frac{1}{1 - \lambda A} - 1\right) v\rangle = \lambda^{-1}\left(\langle v, J_\lambda v\rangle - \|v\|^2\right) \le 0$$

where the last inequality follows from $\|J_\lambda\| \le 1$.
Taking the limit $\lambda \searrow 0$ and by $J_\lambda v \to v$ we get $\langle A^*v, v\rangle \le 0$ and so A^* is dissipative by Lemma 3.10.

Let us now suppose that A and A^* are dissipative, that $G(A)$ is closed and $D(A)$ dense. We have to show that A is m–dissipative. As we argued above $G(A)$ closed is equivalent to $G(1 - A)$ is closed.
Since $G(1 - A)$ is closed and A is dissipative we conclude that $R(1 - A)$ is closed. Indeed, if $\{(1 - A)x_n\}$ is a Cauchy sequence in $R(1 - A)$ then $\{x_n\}$ is a Cauchy sequence $D(A)$. This follows by the fact that A is dissipative which yields $\|(1 - A)(x_n - x_m)\| \ge \|x_n - x_m\|$. Then $(x_n, (1 - A)x_n)$ is a Cauchy sequence in $G(1 - A)$ which hence converges in $G(1 - A)$. So $\{(1 - A)x_n\}$ converges in $R(1 - A)$.
Since we know now that $R(1 - A) = \overline{R(1 - A)}$ we need to show that $\overline{R(1 - A)} = X$. If $\overline{R(1 - A)} \subsetneq X$ then there is a non zero $v \in R(1 - A)^\perp$. Then

$$\langle v, u\rangle = \langle v, Au\rangle \text{ for every } u \in D(A).$$

But this implies that $v \in D(A^*)$ with $A^*v = v$. Then $(1 - A^*)v = 0$. But since A^* is dissipative this implies $v = 0$ and we get a contradiction. So $R(1 - A) = X$ and so A is m–dissipative. □

Example 3.15. Let $X = L^2([0,1])$ and let $A = \frac{d}{dx}$ with

$$D(A) = \{u \in H^1((0,1)) : u(1) = 0\}.$$

For $u, v \in H^1((0,1))$ we have

$$\int_0^1 u'(x)v(x)dx = u(1)v(1) - u(0)v(0) - \int_0^1 u(x)v'(x)dx, \tag{3.5}$$

see [1] p.215. Notice that for $u = v \in D(A)$ we have

$$\int_0^1 u'(x)u(x)dx = -2^{-1}|u(0)|^2 \leq 0.$$

So A is dissipative.
Let now $v \in D(A^*)$. First of all, for $u \in C_0^\infty((0,1))$ from $\langle u', v\rangle = \langle u, A^*v\rangle$ it follows that in the sense of distributions $v' = -A^*v \in L^2([0,1])$. Hence for all $v \in D(A^*)$ we have $v \in H^1((0,1))$ and $A^*v = -v'$. From (3.5) we obtain $u(0)v(0) = 0$ for all $u \in D(A)$. This implies $v(0) = 0$. Viceversa, given any $v \in H^1((0,1))$ with $v(0) = 0$ then from (3.5) we obtain $v \in D(A^*)$.
So $A^* = -\frac{d}{dx}$ with

$$D(A^*) = \{v \in H^1((0,1)) : v(0) = 0\}.$$

From (3.5) we can see that for $u = v \in D(A^*)$ we have

$$\int_0^1 (-v'(x))v(x)dx = -2^{-1}|v(1)|^2 \leq 0.$$

So A^* is dissipative.
It is easy to understand that A is the adjoint of A^*. Hence $G(A)$ is closed. So A is m–dissipative.

4 Extrapolation

Proposition 4.1. *Let A be an m–dissipative operator on a Banach space X with dense domain $D(A)$. There exists a Banach space $\overline{X}$ and an m–dissipative operator $\overline{A}$ on $\overline{X}$ s.t.*

(1) $X \hookrightarrow \overline{X}$ with dense image;

(2) For all $u \in X$ we have $\|u\|_{\overline{X}} = \|J_1 u\|_X$;

(3) $D(\overline{A}) = X$ and the norms $\| \ \|_{D(\overline{A})} := \| \cdot \|_{\overline{X}} + \|\overline{A} \cdot \|_{\overline{X}}$ and $\| \ \|_X$ are equivalent;

(4) $\overline{A}u = Au$ for any $u \in D(A)$.

The pair $(\overline{X}, \overline{A})$ is unique up to isomorphism.

Example 4.2. Take $A = \triangle$ in $X = L^2(\mathbb{R}^n, \mathbb{C})$ with $D(\triangle) = H^2(\mathbb{R}^n, \mathbb{C})$. Set

$$\overline{X} = \{u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}) : \mathcal{F}^* \left[(1 + |\xi|^2)^{-1}\widehat{u}(\xi)\right] \in L^2(\mathbb{R}^n, \mathbb{C})\} \text{ with } \|u\|_{\overline{X}} := \|\mathcal{F}^* \left[(1 + |\xi|^2)^{-1}\widehat{u}(\xi)\right]\|_X.$$

Notice that for $u \in X$ we have $\mathcal{F}^* \left[(1 + |\xi|^2)^{-1}\widehat{u}(\xi)\right] = J_1 u$ and that $\overline{X} = H^{-2}(\mathbb{R}^n, \mathbb{C})$ where for any $s \in \mathbb{R}$

$$H^s(\mathbb{R}^n, \mathbb{C}) := \{f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}) : (1 + |\xi|^2)^{s/2}\widehat{u}(\xi) \in L^2(\mathbb{R}^n, \mathbb{C})\}. \tag{4.1}$$

$L^2(\mathbb{R}^n, \mathbb{C})$ is dense in $H^{-2}(\mathbb{R}^n, \mathbb{C})$.
For any $u \in L^2(\mathbb{R}^n, \mathbb{C})$ the distribution $\triangle u$ belongs to $H^{-2}(\mathbb{R}^n, \mathbb{C})$. So let us define $\overline{A} = \triangle$ with $D(\overline{A}) = L^2(\mathbb{R}^n, \mathbb{C})$. For $u \in D(\overline{A})$

$$\|u\|_{D(\overline{A})} := \|u\|_{\overline{X}} + \|\overline{A}u\|_{\overline{X}} = \|(1 + |\xi|^2)^{-1}\widehat{u}\|_{L^2} + \|(1 + |\xi|^2)^{-1}|\xi|^2\widehat{u}\|_{L^2} \leq 2\|\widehat{u}\|_{L^2} = 2\|u\|_{L^2}$$

and

$$\begin{aligned}
\|u\|_{D(\overline{A})} &:= \|u\|_{\overline{X}} + \|\overline{A}u\|_{\overline{X}} = \|(1 + |\xi|^2)^{-1}\widehat{u}\|_{L^2} + \|(1 + |\xi|^2)^{-1}|\xi|^2\widehat{u}\|_{L^2} \\
&\geq \|(1 + |\xi|^2)^{-1}\widehat{u}\|_{L^2(|\xi|\leq 1)} + \|(1 + |\xi|^2)^{-1}|\xi|^2\widehat{u}\|_{L^2(|\xi|\geq 1)} \\
&= (\int_{|\xi|\leq 1} |(1 + |\xi|^2)^{-1}\widehat{u}|^2)^{\frac{1}{2}} + (\int_{|\xi|\geq 1} |(1 + |\xi|^2)^{-1}|\xi|^2\widehat{u}|^2)^{\frac{1}{2}} \\
&\geq 2^{-1}(\int_{|\xi|\leq 1} |\widehat{u}|^2)^{\frac{1}{2}} + 2^{-1}(\int_{|\xi|\geq 1} |\widehat{u}|^2)^{\frac{1}{2}} \geq 2^{-2}\|\widehat{u}\|_{L^2}.
\end{aligned}$$

Proof of Prop. 4.1. For $u \in X$ we consider $|||u||| := \|J_1 u\|_X$. This is a norm on X. We denote by $\overline{X}$ the completion of X by this norm, which is unique up to isomorphism and is s.t. (1) holds. Set $\| \|_{\overline{X}} = ||| \, |||$. We have

$$J_1 A u = (1 - A)^{-1}[1 + (A - 1)]u = J_1 u - u \quad \forall\, u \in D(A).$$

Then for $u \in D(A)$

$$\|Au\|_{\overline{X}} = \|J_1 A u\|_X \leq \|J_1 u\|_X + \|u\|_X = 2\|u\|_X.$$

So A can be extended into an operator $\widetilde{A} \in \mathcal{L}(X, \overline{X})$, in a unique way since $D(A)$ is dense in X. We set $\overline{A} = \widetilde{A}$ with $D(\overline{A}) = X$.
Turning to claim (3), for $u \in D(A)$ we have

$$\|u\|_{D(\overline{A})} = \|u\|_{\overline{X}} + \|\overline{A}u\|_{\overline{X}} \leq \|u\|_{\overline{X}} + 2\|u\|_X \leq 3\|u\|_X.$$

Notice that since for any $u \in X$ exists $D(A) \ni u_n \to u$ in X and since $\overline{A} \in \mathcal{L}(X, \overline{X})$ then $\|u\|_{D(\overline{A})} \leq 3\|u\|_X$ remains true for all $u \in X$.
For $u \in D(A)$ by the triangular inequality we have

$$\|u\|_{D(\overline{A})} = \|u\|_{\overline{X}} + \|\overline{A}u\|_{\overline{X}} = \|J_1 u\|_X + \|J_1 A u\|_X = \|J_1 u\|_X + \|J_1 u - u\|_X \geq \|u\|_X$$

and by continuity $\|u\|_{D(\overline{A})} \geq \|u\|_X$ remains true for all $u \in X$.

Claim (4), that is $\overline{A}u = Au$ for any $u \in D(A)$, holds by construction.
We now check that $\overline{A}$ is m–dissipative. Let $\lambda > 0$. For $u \in D(A)$ and $v = J_1 u$ we have

$$v - \lambda A v = (J_1 - \lambda A J_1)u = (J_1 - \lambda J_1 A)u = J_1(1 - \lambda A)u.$$

Since A is dissipative $\forall\ u \in D(A)$

$$\|u - \lambda A u\|_{\overline{X}} := \|J_1(1-\lambda A)u\|_X = \|v - \lambda A v\|_X \geq \|v\|_X = \|J_1 u\|_X =: \|u\|_{\overline{X}} \Rightarrow \|u - \lambda A u\|_{\overline{X}} \geq \|u\|_{\overline{X}}.$$

For $u \in X \backslash D(A)$ let $(u_n)_n$ be a sequence in $D(A)$ with $u_n \to u$ in X. Then by $\widetilde{A} = A$ in $D(A)$

$$\|u_n - \lambda \widetilde{A} u_n\|_{\overline{X}} \geq \|u_n\|_{\overline{X}} \Rightarrow \|u - \lambda \widetilde{A} u\|_{\overline{X}} \geq \|u\|_{\overline{X}}$$

where we used $\widetilde{A} \in \mathcal{L}(X, \overline{X})$. This implies $\|u - \lambda \overline{A} u\|_{\overline{X}} \geq \|u\|_{\overline{X}}$ for any $u \in D(\overline{A})$ and so $\overline{A}$ is dissipative.
We next show that for any $f \in \overline{X}$ there exists $u \in X$ s.t. $f = u - \widetilde{A}u$. We consider a sequence (f_n) in X s.t. $f_n \to f$ in $\overline{X}$. Consider the sequence (u_n) in $D(A)$ defined by $u_n := J_1 f_n$. Since $\|u_n - u_m\|_X = \|f_n - f_m\|_{\overline{X}}$ and since (f_n) is Cauchy in $\overline{X}$ then (u_n) is Cauchy in X. Then u_n converges in X to an $u \in X$. Then by $\widetilde{A} \in \mathcal{L}(X, \overline{X})$

$$f_n = (1 - A)u_n = (1 - \widetilde{A})u_n \quad \forall\ n \Rightarrow f = (1 - \widetilde{A})u = (1 - \overline{A})u.$$

□

Corollary 4.3. *If $x \in X$ is s.t. $\overline{A}x \in X$ then $x \in D(A)$ and $\overline{A}x = Ax$.*

Proof. Let $f = x - \overline{A}x \in X$. Since A is m–dissipative there exists $u \in D(A)$ s.t. $f = u - Au$. Hence $(x - u) - \overline{A}(x - u) = 0$. Since $\overline{A}$ is dissipative this implies $x = u$. □

5 Contraction semigroups

Let A be m–dissipative in X with $\overline{D(A)} = X$. Let $A_\lambda := J_\lambda A$. Notice that $A_\lambda = \lambda^{-1}(J_\lambda - 1)$ since

$$\begin{aligned}\lambda^{-1}(J_\lambda - 1) &= \lambda^{-1}\left((1 - \lambda A)^{-1} - (1 - \lambda A)(1 - \lambda A)^{-1}\right) \\ &= \lambda^{-1}\left(1 - (1 - \lambda A)\right)(1 - \lambda A)^{-1} = A(1 - \lambda A)^{-1} = A_\lambda.\end{aligned}$$

Then $\|A_\lambda\| \leq 2\lambda^{-1}$ and we can set $T_\lambda(t) = e^{tA_\lambda}$ where , as for any bounded operator,

$$e^{tA_\lambda} = \sum_{n=0}^{\infty} \frac{(tA_\lambda)^n}{n!}.$$

Theorem 5.1. *For any $x \in X$ we have that $\lim_{\lambda \searrow 0} T_\lambda(t)x$ converges uniformly on compact sets to a function $u(t) \in C([0, \infty), X)$. We set $T(t)x := u(t)$. Then*

$$\begin{aligned} &T(t) \in \mathcal{L}(X) \text{ with } \|T(t)\| \leq 1 \text{ for all } t \geq 1 \\ &T(0) = I \\ &T(t + s) = T(t)T(s) \text{ for all } t, s. \end{aligned} \tag{5.1}$$

If $x \in D(A)$ then $u(t) = T(t)x$ is the unique solution of the following problem:

$$\begin{aligned} & u \in C([0,\infty), D(A)) \cap C^1((0,\infty), X) \\ & u'(t) = Au(t) \text{ for all } t > 0 \\ & u(0) = x, \end{aligned} \tag{5.2}$$

where we endow $D(A)$ with the norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$.

For all $x \in D(A)$ we have $T(t)A = AT(t)x$.

Proof. There are 5 steps in the proof.
Step 1. We claim that if we set $u_\lambda(t) := T_\lambda(t)x$, we have always $\|u_\lambda(t)\| \le \|x\|$. Notice that later, when we prove that $\lim_{\lambda \searrow 0} u_\lambda(t) = u(t)$, this implies immediately that $\|u(t)\| \le \|x\|$. By $A_\lambda = \lambda^{-1}(J_\lambda - 1)$ we have $T_\lambda(t) = e^{tA_\lambda} = e^{t\lambda^{-1}(J_\lambda - 1)} = e^{t\lambda^{-1}J_\lambda}e^{-t\lambda^{-1}}$ and $\|T_\lambda(t)\| \le e^{t\lambda^{-1}\|J_\lambda\|}e^{-t\lambda^{-1}} \le 1$ by $\|J_\lambda\| \le 1$.

Step 2. Let $x \in D(A)$. We claim that the family of functions $(u_\lambda(t))_{\lambda>0}$, which is contained in $C([0,\infty), X)$, for $\lambda \searrow 0$ converges uniformly on compact sets to a function $u(t) \in C([0,\infty), X)$. We set $T(t)x = u(t)$. It is elementary to then show that $T(t) : D(A) \to X$ is a linear operator such that $\|T(t)x\|_X \le \|x\|_X$. that can be extended in a unique way to an operator $T(t) \in \mathcal{L}(X)$ s.t.$\|T(t)\| \le 1$.
To prove the claim, we make another claim. This 2nd claim states that A_λ and A_μ commute for any pair in $\lambda, \mu \in \mathbb{R}_+$. One expresses this writing $[A_\lambda, A_\mu] = 0$, where $[T, S] := TS - ST$. We will prove $[A_\lambda, A_\mu] = 0$ in a moment. Recall that if T and S are two bounded operators with $[T, S] = 0$ then $e^{T+S} = e^T e^S$. Then $[A_\lambda, A_\mu] = 0$ implies

$$\frac{d}{ds}(T_\lambda(st)T_\mu(t - st))x = \frac{d}{ds}e^{stA_\lambda + (t-st)A_\mu}x = tT_\lambda(st)T_\mu(t - st)(A_\lambda - A_\mu)x$$

This implies

$$\begin{aligned} & \|u_\lambda(t) - u_\mu(t)\| = \|T_\lambda(t)x - T_\mu(t)x\| = \|\int_0^1 \frac{d}{ds}(T_\lambda(st)T_\mu(t - st))x\| \\ & \le t\int_0^1 \|T_\lambda(st)T_\mu(t - st)(A_\lambda - A_\mu)x\| \le t\|(A_\lambda - A_\mu)x\| = t\|(J_\lambda - J_\mu)Ax\|. \end{aligned}$$

This means that since $J_\lambda Ax \overset{\lambda \searrow 0}{\to} Ax$, our claim stated at the beginning of this step is true.
Turning to $[A_\lambda, A_\mu] = 0$, this follows from

$$A_\lambda A_\mu = A(1 - \lambda A)^{-1}A(1 - \mu A)^{-1} = (\lambda - \mu)^{-1}A\left((1 - \lambda A)^{-1} - (1 - \mu A)^{-1}\right) = A_\mu A_\lambda$$

where we used

$$\begin{aligned} & (1 - \lambda A)^{-1} - (1 - \mu A)^{-1} = (1 - \lambda A)^{-1}(1 - \mu A)(1 - \mu A)^{-1} - (1 - \lambda A)^{-1}(1 - \lambda A)(1 - \mu A)^{-1} \\ & = (\lambda - \mu)(1 - \lambda A)^{-1}A(1 - \mu A)^{-1}. \end{aligned}$$

Step 3. Since $\overline{D(A)} = X$, step 2 implies a unique extension $T(t) : X \to X$ and we have $\|T(t)\| \leq 1$.

We check now that for any $x \in X$ we have $\lim_{\lambda \searrow 0} T_\lambda(t)x = T(t)x$ uniformly on compact sets in $C([0,\infty), X)$. We know already that this is true for $x \in D(A)$ and by a density argument we claim it holds also for $x \notin D(A)$. Since $\overline{D(A)} = X$ we can consider a sequence $(x_n)_n$ in $D(A)$ with $x_n \to x$ in X. Then

$$\begin{aligned}
&\|T_\lambda(t)x - T(t)x\| \leq \|T_\lambda(t)(x - x_n)\| + \|T(t)(x_n - x)\| + \|T_\lambda(t)x_n - T(t)x_n\| \\
&\leq 2\|x - x_n\| + \|T_\lambda(t)x_n - T(t)x_n\|
\end{aligned}$$

immediately yields $\lim_{\lambda \searrow 0} T_\lambda(t)x = T(t)x$ uniformly on compact intervals.
From $T_\lambda(t)T_\lambda(s) = T_\lambda(t+s)$ we get $T(t)T(s) = T(t+s)$. Indeed, for any $x \in X$ we have

$$\begin{aligned}
&\|T(t)T(s)x - T(t+s)x\| \leq \|T(t)T(s)x - T(t)T_\lambda(s)x\| + \|T(t)T_\lambda(s)x - T_\lambda(t)T_\lambda(s)x\| \\
&+ \|\underbrace{T_\lambda(t)T_\lambda(s)}_{T_\lambda(t+s)} x - T(t+s)x\| \to 0 \text{ as } \lambda \searrow 0.
\end{aligned}$$

Step 4. Let $x \in D(A)$ and and consider $u'_\lambda(t) = T_\lambda(t)A_\lambda x = A_\lambda T_\lambda(t)x$. Then

$$\begin{aligned}
&\|u'_\lambda(t) - T(t)Ax\| = \|T_\lambda(t)A_\lambda x - T(t)Ax\| \leq \|T_\lambda(t)Ax - T(t)Ax\| + \|T_\lambda(t)(A_\lambda x - Ax)\| \\
&\leq \|T_\lambda(t)Ax - T(t)Ax\| + \|A_\lambda x - Ax\|.
\end{aligned}$$

Hence we have

$$\lim_{\lambda \searrow 0} u'_\lambda(t) = \lim_{\lambda \searrow 0} T_\lambda(t)A_\lambda x = T(t)Ax \text{ uniformly on compact sets in } C([0,\infty), X).$$

Then taking $\lambda \searrow 0$ on both sides, we get

$$u_\lambda(t) = x + \int_0^t u'_\lambda(s)ds \to T(t)x = x + \int_0^t T(s)Axds,$$

from which we see that for $x \in D(A)$ we have $T(t)x \in C^1([0,\infty), X)$ with derivative $T(t)Ax$. We now prove $AT(t)x = T(t)Ax$ for $x \in D(A)$. We have $u'_\lambda(t) = A_\lambda T_\lambda(t)x = AJ_\lambda T_\lambda(t)x$. We claim that $\lim_{\searrow} J_\lambda T_\lambda(t)x = T(t)x$ in the topology of uniform convergence on compact sets in $C([0,\infty), X)$. To prove this claim we write

$$\begin{aligned}
&\|J_\lambda T_\lambda(t)x - T(t)x\| = \|T_\lambda(t)x - T(t)x\| + \|T_\lambda(t)x - J_\lambda T_\lambda(t)x\| \\
&= \|T_\lambda(t)x - T(t)x\| + \|x - T_\lambda(t)J_\lambda x\| \leq \|T_\lambda(t)x - T(t)x\| + \|x - J_\lambda x\|.
\end{aligned}$$

But now we know that as $\lambda \searrow 0$ the r.h.s. converges to 0 in compact subsets of $[0,\infty)$. This yields the desired claim. So, summing up, we have

$$\lim_{\lambda \searrow 0} (J_\lambda T_\lambda(t)x, \underbrace{AJ_\lambda T_\lambda(t)x}_{u'_\lambda(t)}) = (T(t)x, T(t)Ax) \text{ in } X \times X.$$

Since A is m–dissipative it follows that its graph $G(A)$ is closed $X \times X$ (recall that $G(A)$ is closed iff $G(1-A)$ is closed and this is closed because $G(J_1)$ is closed). This means that $T(t)x \in D(A)$ with $AT(t)x = T(t)Ax$.

Now we are in position to show that $T(t)x \in C([0,\infty), D(A))$ with $D(A)$ endowed with the norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$ for $x \in D(A)$. First of all we have $T(t)x \in C([0,\infty), X)$ and second we have $AT(t)x = T(t)Ax \in C([0,\infty), X)$. This yields the desired claim.

Step 5. We check the uniqueness of the solution of (5.2). Let $u(t)$ be a solution of (5.2) and set $v(t) := T(\tau - t)u(t)$ for $\tau > 0$ and $t \in [0,\tau]$. We have $v(t) \in C([0,\tau], D(A)) \cap C^1([0,\tau], X)$ and, in particular, from the chain rule and the product rule we have

$$v'(t) = -AT(\tau - t)u(t) + T(\tau - t)u'(t) = -T(\tau - t)Au(t) + T(\tau - t)Au(t) = 0.$$

So in particular for any $x' \in X'$, we have that $\langle v(t), x' \rangle \in C^0([0,\tau], \mathbb{R})$, is differentiable in $(0,\tau)$ with $\frac{d}{dt}\langle v(t), x' \rangle = \langle \frac{d}{dt} v(t), x' \rangle = 0$. Then by Lagrange's Theorem we have that $\langle v(t), x' \rangle$ is constant, in particular with $\langle v(0), x' \rangle = \langle v(\tau), x' \rangle$ for any $x' \in X'$, and so $v(0) = v(\tau)$ in X. So $u(\tau) = v(\tau) = v(0) = T(\tau)x$. So for any $\tau \geq 0$ we get $u(\tau) = T(\tau)x$. □

Definition 5.2 (Contraction semigroup)**.** A family $(T(t))_{t \geq 0} \in \mathcal{L}(X)$ is a contraction semigroup if the following happens.

(1) $\|T(t)\| \leq 1$ for all $t \geq 0$.

(2) $T(0) = I$

(3) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$.

(4) For any $x \in X$ we have $T(t)x \in C^0([0,\infty), X)$.

If instead only the conditions (2)–(4) are satisfied, then $(T(t))_{t \geq 0}$ is called C_0–semigroup.

Notice that a special case of the above definition is the following.

Definition 5.3 (Isometry group)**.** A family $(T(t))_{t \in \mathbb{R}} \in \mathcal{L}(X)$ is an isometry group if the following happens.

(1) $\|T(t)x\| = \|x\|$ for all $t \in \mathbb{R}$ and all $x \in X$.

(2) $T(0) = I$

(3) $T(t)T(s) = T(t+s)$ for all $t, s \in \mathbb{R}$.

(4) For any $x \in X$ we have $T(t)x \in C^0(\mathbb{R}, X)$.

Definition 5.4 (Generator of a contraction semigroup)**.** Is the operator L defined by

$$D(L) = \{x \in X : \lim_{h \searrow 0} \frac{T(h)x - x}{h} \text{ exists in } X \}$$

and for $x \in D(L)$

$$Lx := \lim_{h \searrow 0} \frac{T(h)x - x}{h}.$$

We have the following examples :

1. $\frac{d}{dx}$ in $L^p(\mathbb{R}, \mathbb{R})$ for $p \in [1, \infty)$ with $D(\frac{d}{dx}) = W^{1,p}(\mathbb{R}, \mathbb{R})$ and $T(t)u(x) = u(x+t)$.
2. $-\frac{d}{dx}$ in $L^p(\mathbb{R}, \mathbb{R})$ for $p \in [1, \infty)$ with $D(\frac{d}{dx}) = W^{1,p}(\mathbb{R}, \mathbb{R})$ and $T(t)u(x) = u(x-t)$.
3. $\frac{d}{dx}$ in $L^p(\mathbb{R}_+, \mathbb{R})$ for $p \in [1, \infty)$ with $D(\frac{d}{dx}) = W^{1,p}(\mathbb{R}_+, \mathbb{R})$ and $T(t)u(x) = u(x+t)$.
4. $-\frac{d}{dx}$ in $L^p(\mathbb{R}_+, \mathbb{R})$ for $p \in [1, \infty)$ with $D(\frac{d}{dx}) = \{u \in W^{1,p}(\mathbb{R}_+, \mathbb{R}) : u(0) = 0\}$ and $T(t)u(x) = u(x-t)$ for $x > t$ and $T(t)u(x) = 0$ for $x \leq t$.
5. The operator $A = \frac{d}{dx}$ with $D(A) = \{u \in H^1((0,1)) : u(1) = 0\}$ in $L^2([0,1])$ has corresponding group

   $$T(t)u(x) = \begin{cases} u(x+t) \text{ for } x+t < 1, \\ 0 \text{ for } x+t \geq 1. \end{cases}$$

   Notice that for any u we have $T(t)u = 0$ for $t \geq 1$.

Example 5.5. The following is an isometry group in any $L^p(\mathbb{R}, \mathbb{R})$ for $p \in [1, \infty)$: $T(t)f(x) := f(x-t)$. The only nontrivial condition to check is (4). By density it is enough to consider $f \in C_c^\infty(\mathbb{R}, \mathbb{R})$. Then

$$\|T(h)f(x) - f(x)\|_{L^p_x} = |h| \; \| \int_0^1 f'(x - th)dt\|_{L^p_x} \leq |h| \; \int_0^1 \|f'(\cdot - th)\|_{L^p} dt = |h| \; \|f'\|_{L^p} \to 0$$

as $h \searrow 0$ by Minkowsky inequality.

Then we claim that $L = -\frac{d}{dx}$. If $f \in D(L)$ then for $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} Lf(x)\phi(x)dx &= \lim_{h \searrow 0} \int_{\mathbb{R}} \frac{f(x-h) - f(x)}{h} \phi(x)dx \\ &= \lim_{h \searrow 0} \int_{\mathbb{R}} f(x) \frac{\phi(x+h) - \phi(x)}{h} dx = \int_{\mathbb{R}} f(x)\phi'(x)dx. \end{aligned}$$

So $Lf = -\frac{d}{dx}f \in L^p(\mathbb{R}, \mathbb{R})$. If instead we consider $f \in L^p(\mathbb{R}, \mathbb{R})$ with $\frac{d}{dx}f \in L^p(\mathbb{R}, \mathbb{R})$ we have

$$\frac{T(h)f(x) - f(x)}{h} = -\int_0^1 f'(x - th)dt = -\int_0^1 T(ht)f'(x)dt$$

Then
$$\frac{T(h)f(x)-f(x)}{h}-f'(x)=-\int_0^1 (T(ht)-I)f'(x)dt \to 0 \text{ as } h\searrow 0$$
by condition (4).

Similarly, $T(t)$ defines a contraction semigroup in $C_0(\mathbb{R},\mathbb{R})$ with $L=-\frac{d}{dx}$, that is $D(L)$ coincides with the set of $f\in C_0(\mathbb{R},\mathbb{R})$ s.t. $f'\in C_0(\mathbb{R},\mathbb{R})$.

On the other hand, for $T(t)f$ to be continuous in $L^\infty(\mathbb{R},\mathbb{R})$ it is necessary that f be uniformly continuous and in $L^\infty(\mathbb{R},\mathbb{R})$. So $T(t)$ does not define a contraction semigroup in $L^\infty(\mathbb{R},\mathbb{R})$.

Notice that the difference between all these cases is that $e^{-t\frac{d}{dx}}$ will define a contraction semigroup in $L^p(\mathbb{R},\mathbb{R})$ for $p<\infty$ and in $C_0(\mathbb{R},\mathbb{R})$ because $-\frac{d}{dx}$ is m dissipative with $D(\frac{d}{dx})$ dense. Notice that also $\frac{d}{dx}$ is m dissipative and its corresponding contraction semigroup is $(e^{t\frac{d}{dx}}f)(x)=f(x+t)$.

Example 5.6. Consider $X=\{f\in C_0([0,\infty),\mathbb{R}) : f(0)=0\}$ and for $t\ge 0$ set
$$T(t)f(x):=\begin{cases} f(x-t) \text{ for } x\ge t\\ \quad 0 \text{ for } x\le t\end{cases}$$
Then this is a contraction semigroup in X with $L=-\frac{d}{dx}$.

This is a contraction semigroup also in $L^p([0,\infty))$ for $p\in[1,\infty)$ with $L=-\frac{d}{dx}$ with boundary condition $f(0)=0$. If $f\in D(L)$ then for $\phi\in C_c^\infty(\mathbb{R}_+,\mathbb{R})$ where $\mathbb{R}_+=(0,\infty)$
$$\begin{aligned}\int_{\mathbb{R}_+} Lf(x)\phi(x)dx &= \lim_{h\searrow 0}\int_{\mathbb{R}_+}\frac{f(x-h)-f(x)}{h}\phi(x)dx\\ &= \lim_{h\searrow 0}\int_{\mathbb{R}_+} f(x)\frac{\phi(x+h)-\phi(x)}{h}dx=\int_{\mathbb{R}_+} f(x)\frac{d}{dx}\phi(x)dx.\end{aligned}$$
So $Lf=-\frac{d}{dx}f\in L^p(\mathbb{R}_+,\mathbb{R})$. The limit $f(0^+)$ exists. It has to be equal to 0 since from
$$\frac{T(h)f(x)-f(x)}{h}=-\int_0^1 T(ht)f'(x)dt$$
for $x<h$ we obtain
$$f(x)=h\int_0^1 T(ht)f'(x)dt \Rightarrow f(x)=2x\int_0^{\frac12} f'(x-2xt)dt\to 0 \text{ as } x\searrow 0.$$
Viceversa, let f be continuous with $f(0)=0$ and such that $f^{(j)}\in L^p(\mathbb{R}_+,\mathbb{R})$ for $j=0,1$. Then, setting $g(x)=0$ for $x<0$ and $g(x)=f(x)$ for $x\ge 0$, we get $g^{(j)}\in L^p(\mathbb{R},\mathbb{R})$ for $j=0,1$. Then for $x>0$ we have $T(h)f(x)=g(x-h)$.

Example 5.7. Consider $X = C_0([0,\infty),\mathbb{R})$ and for $t \geq 0$ set

$$T(t)f(x) := \{f(x+t)$$

Then this is a contraction semigroup in X. This is a contraction semigroup also in $L^p([0,\infty))$ for $p \in [1,\infty)$ with $L = \frac{d}{dx}$ with domain $W^{1,p}(\mathbb{R}_+)$.

Proposition 5.8. *Let L be the generator of a contraction semigroup $(T(t))_{t\geq 0} \in \mathcal{L}(X)$. Then L is m–dissipative and $\overline{D(L)} = X$.*

Proof. 1. L is dissipative For $x \in D(L)$, $\lambda > 0$ and $h > 0$ we have

$$\begin{aligned}
&\|x - \lambda\frac{T(h)x - x}{h}x\| = \|(1+\lambda h^{-1})x - \lambda h^{-1}T(h)x\| \geq (1+\lambda h^{-1})\|x\| - \lambda h^{-1}\|T(h)x\| \\
&= \|x\| + \lambda h^{-1}(\|x\| - \|T(h)x\|) \geq \|x\|
\end{aligned}$$

so that taking the limit for $h \searrow 0$ we get $\|x - \lambda Lx\| \geq \|x\|$.

2. L is m–dissipative Given $x \in X$ we need to show that there is $y \in D(L)$ s.t. $(1-L)y = x$. Formally, the idea is to set $y = (1-L)^{-1}x$, which of course makes no sense yet. However, thinking of the Laplace transform which formally gives us

$$(1-L)^{-1}x = \int_0^\infty e^{-t}e^{tL}xdt$$

we define y as the r.h.s. (which makes perfect sense) of the above equality. Then

$$\begin{aligned}
&\frac{T(h)y - y}{h} = h^{-1}\int_0^\infty e^{-t}(T(t+h) - T(t))xdt \\
&= h^{-1}\int_h^\infty e^{-(t-h)}T(t)xdt - h^{-1}\int_h^\infty e^{-t}T(t)xdt - h^{-1}\int_0^h e^{-t}T(t)xdt \\
&= \frac{e^h - 1}{h}\int_h^\infty e^{-t}T(t)xdt - h^{-1}\int_0^h e^{-t}T(t)xdt \to y - x \text{ as } h \searrow 0.
\end{aligned}$$

This means that $y \in D(L)$ with $Ly = y - x$ or, equivalently, $(1-L)y = x$. Hence L is m–dissipative.

Remark 5.9. So we have proved that if we set $Jx = \int_0^\infty e^{-t}e^{tL}xdt$ then $Jx \in D(L)$.

Notice on the other hand that if e^{tL} is a group of isometries and if $x \notin D(L)$ then $e^{tL}x \notin D(L)$ for all $t \in \mathbb{R}$. Nonetheless, $Jx \in D(L)$.

The fact about $e^{tL}x \notin D(L)$ can be seen noticing that if $T(t)$ is a contraction semigroup with generator A and if $x_0 \in D(A)$ then $T(t)x_0 \in C([0,\infty), D(A))$. So, if $e^{t_0L}x \in D(L)$ for some $t_0 > 0$, for example, then for $A = -L$ and using $D(A) = D(L)$ we have that $e^{t_0L}x \in D(L) = D(A)$ implies $x = T(t_0)e^{t_0L}x \in D(A) = D(L)$, which contradicts $x \notin D(L)$. So $e^{tL}x \notin D(L)$ for all $t \in \mathbb{R}$.

3. $D(L)$ is dense in X We set $x_t = t^{-1}\int_0^t T(s)xds$. Then $x_t \to x$ as $t \searrow 0$ by the continuity of $T(t)$. We show that $x_t \in D(L)$ for $t > 0$. Of course, as we will see in a moment, this will be a simple computation, but the heuristic idea could be encapsulated in the following formal integration:

$$Ltx_t = L\int_0^t e^{sL}ds\; x = (e^{tL} - 1)x. \tag{5.3}$$

The rigorous argument is as follows and yields $Ltx_t = e^{tL}x - x$:

$$\begin{aligned}
&\frac{T(h)x_t - x_t}{h} = h^{-1}t^{-1}\int_h^{t+h} T(s)xds - h^{-1}t^{-1}\int_0^t T(s)xds \\
&= h^{-1}t^{-1}t^{-1}\int_h^t T(s)xds + h^{-1}t^{-1}\int_t^{t+h} T(s)xds - h^{-1}t^{-1}\int_h^t T(s)xds - h^{-1}\int_0^h T(s)xds \\
&= h^{-1}t^{-1}\int_t^{t+h} T(s)xds - h^{-1}t^{-1}\int_0^h T(s)xds \to t^{-1}T(t)x - t^{-1}x \text{ as } h \searrow 0.
\end{aligned}$$

So $x_t \in D(L)$ with $Ltx_t = t^{-1}T(t)x - t^{-1}x$, confirming the formal computation (5.4).

$$Ltx_t = L\int_0^t e^{sL}ds\; x = (e^{tL} - 1)x. \tag{5.4}$$

The rigorous argument is as follows and yields $Ltx_t = e^{tL}x - x$:

$$\begin{aligned}
&\frac{T(h)x_t - x_t}{h} = h^{-1}t^{-1}\int_h^{t+h} T(s)xds - h^{-1}t^{-1}\int_0^t T(s)xds \\
&= h^{-1}t^{-1}t^{-1}\int_h^t T(s)xds + h^{-1}t^{-1}\int_t^{t+h} T(s)xds - h^{-1}t^{-1}\int_h^t T(s)xds - h^{-1}\int_0^h T(s)xds \\
&= h^{-1}t^{-1}\int_t^{t+h} T(s)xds - h^{-1}t^{-1}\int_0^h T(s)xds \to t^{-1}T(t)x - t^{-1}x \text{ as } h \searrow 0.
\end{aligned}$$

So $x_t \in D(L)$ with $Ltx_t = t^{-1}T(t)x - t^{-1}x$, confirming the formal computation (5.4).

□

Example 5.10. In $X = L^2([0,1])$ the following is a contraction semigroup

$$T(t)u(x) = \begin{cases} u(x+t) \text{ for } x + t < 1, \\ 0 \text{ for } x + t \geq 1. \end{cases}$$

Let L be the generator. Then for $x < 1$ we have

$$\lim_{t\to 0^+} \frac{T(t)u(x) - u(x)}{t} = \lim_{t\to 0^+} \frac{u(x+t) - u(x)}{t} = Lu(x)$$

implies that $Lu(x) = u'(x)$. So the derivative exists a.e. and equals $Lu(x)$. In fact this is also an equality in a distributional sense as can be seen using test functions from $C_c^\infty((0,1))$.

From the definition, for any $u \in L^2([0,1])$ we have $(T(t)u)(1) = 0$. For $u \in D(L)$ we know that $T(t)u \in C([0,\infty), D(L))$. So in particular, since $D(L) \subset H^1((0,1)) \subset C^0([0,1])$ it follows $T(t)u \in C([0,\infty), C^0([0,1])$ and so $(T(t)u)(1) = \text{ev}_1 \circ T(t)u \in C^0([0,1])$ where $\text{ev}_{s_0} : C^0([0,1]) \to \mathbb{R}$ is the map $\text{ev}_{s_0} f := f(s_0)$, defined for any preassigned $s_0 \in [0,1]$. So

$$u(1) = \lim_{t \searrow 0} (T(t)u)(1) = 0.$$

This means that $G(L) \subseteq G(A)$ for A the operator in Example 3.15. Suppose now that they are not equal and let $f \in D(A) \backslash D(L)$. Set $F = (1-A)f$. Since L is m–dissipative we have let $g \in D(L)$ s.t. $F = (1-L)g = (1-A)g$. Then $(1-A)(f-g) = 0$. Since A is dissipative we have $f = g$.

Theorem 5.11. *A is the generator of a contraction semigroup in X if and only if A is m–dissipative with dense domain.*

Proof. If A is the generator of a contraction semigroup in X then A is m–dissipative with dense domain by Prop. 5.8.

Viceversa, let A be m–dissipative with dense domain. By Theorem 5.1 it remains defined a contraction semigroup $(T(t))_{t \geq 0}$. This has a generator L. We show now that $L = A$.

For $x \in D(A)$ recall that then $u(t) := T(t)x$ satisfies (5.2), that is, it is the unique solution of the following problem:

$$\begin{aligned}
&u \in C([0,\infty), D(A)) \cap C^1((0,\infty), X) \\
&u'(t) = Au(t) \text{ for all } t > 0 \\
&u(0) = x,
\end{aligned}$$

Then for $h > 0$ we have

$$T(h)x = x + \int_0^h T(t)Axdt \Rightarrow \lim_{h \searrow 0} \frac{T(h)x - x}{h} = \lim_{h \searrow 0} h^{-1} \int_0^h T(t)Axdt = Ax.$$

Then $x \in D(L)$ with $Lx = Ax$. So $G(A) \subseteq G(L)$.

Let $y \in D(L)$. Since A is m–dissipative there exists $x \in D(A)$ s.t. $x - Ax = y - Ly$. Since $G(A) \subseteq G(L)$ we have $Ax = Lx$ and so $(x-y) - L(x-y) = 0$. Since by Prop. 5.8 L is m– dissipative, and so in particular dissipative, we have $x = y$ and so $A = L$.

□

5.1 Self–adjoint ≤ 0 operators in Hilbert spaces

In the case of self–adjoint negative operators in Hilbert spaces things are simpler thanks to the following formulation of the Spectral Theorem for separable Hilbert spaces, which can be viewed as a *diagonalization* theorem. For a proof see Ch. 8 [6].

Theorem 5.12 (Spectral Theorem for separable Hilbert spaces)**.** *Let X be a separable Hilbert space and let A be self–adjoint. Then there exists a measure space (Ω, μ), an isometric isomorphism $U : L^2(\Omega, \mu) \to X$ and a real valued measurable function $a(\omega)$ in Ω s.t.*

$$U^{-1}AUf(\omega) = a(\omega)f(\omega) \text{ for any } Uf \in D(A).$$

Given $f \in L^2(\Omega, \mu)$ we have $Uf \in D(A)$ if and only if $a(\omega)f(\omega) \in L^2(\Omega, \mu)$.

□

Now we consider a self–adjoint operator $A \leq 0$ on a separable Hilbert space and its corresponding contraction semigroup $T(t)$.

Theorem 5.13. *Let X be a e Hilbert space, assume that A is self–adjoint ≤ 0. Let $x \in X$ and let $u(t) = T(t)x$. Then $u(t)$ is the unique solution of the following problem:*

$$u \in C^0([0, \infty), X) \cap C^0((0, \infty), D(A)) \cap C^1((0, \infty), X) \tag{5.5}$$

$$u'(t) = Au(t) \text{ for all } t > 0 \tag{5.6}$$

$$u(0) = x \tag{5.7}$$

We also have

$$\|Au(t)\| \leq \frac{\|x\|}{t\sqrt{2}} \tag{5.8}$$

$$-\langle Au(t), u(t)\rangle \leq \frac{\|x\|^2}{2t}. \tag{5.9}$$

Finally, if $x \in D(A)$ we have

$$\|Au(t)\|^2 \leq -\frac{1}{2t}\langle Ax, x\rangle. \tag{5.10}$$

Proof. We will consider only the case when the space X is separable. Then the Spectral Theorem allows us to reduce to the special case in which $X = L^2(\Omega, \mu)$ and $Au(\omega) = a(\omega)u(\omega)$ for a real valued measurable function $a(\omega) \leq 0$. As a solution of (5.6)–(5.7) the only possible candidate is

$$u(t, \omega) = e^{ta(\omega)}x(\omega). \tag{5.11}$$

Notice that for $t > 0$ and any $n \in \mathbb{N}$ the function $f_t(a) = a^n e^{-ta}$ for $a \in \mathbb{R}_+$ has a point of maximum at $a_M = \frac{n}{t}$ with maximum value $f_t(a_M) = \frac{n^n}{t^n}e^{-n}$. This implies that $u(t) \in D(A^n)$ for any $t > 0$. In particular this gives for $n = 1$

$$\|Au(t)\| \leq \|x\| \sup_{a<0} |ae^{ta}| \leq \frac{\|x\|}{te} < \frac{\|x\|}{t\sqrt{2}} \tag{5.12}$$

and

$$-\langle Au(t), u(t)\rangle \leq \|Au(t)\| \; \|u(t)\| \leq \frac{\|x\|^2}{et} \leq \frac{\|x\|^2}{2t}. \tag{5.13}$$

which imply (5.8)–(5.9). If $x \in D(A)$ we have

$$\|Au(t)\|^2 = \langle A^2u(t), u(t)\rangle \le \|ae^{ta}\sqrt{|a|}x\|\|\sqrt{|a|}x\| \le \frac{1}{et}\|\sqrt{|a|}x\|^2 = -\frac{1}{et}\langle Ax, x\rangle.$$

This implies (5.10).
Finally the (5.11) satisfies (5.5) and more generally $u \in C^l((0,\infty), D(A^n))$ for all l, n.

5.2 The semigroup $e^{t\triangle}$

We set $K_t(x) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. We know that in $L^2(\mathbb{R}^n, \mathbb{C})$ the operator $\triangle$ is m–dissipative and that $e^{t\triangle} f = K_t * f$ for any $f \in L^2(\mathbb{R}^n, \mathbb{R})$.
Let now $p \in [1,\infty)$ with $p \neq 2$ and set $T(t)f = K_t * f$ for any $f \in L^p(\mathbb{R}^n, \mathbb{C})$ and any $t > 0$. Set $T(0) = I$. Using the Fourier transform

$$\begin{aligned}\mathcal{F}(K_{t+s} * f) &= e^{-t|\xi|^2} e^{-s|\xi|^2} \widehat{f} = (2\pi)^{-\frac{n}{2}} \mathcal{F}\left(\underbrace{\mathcal{F}^*(e^{-t|\xi|^2})}_{(2t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}} * (K_s * f) \right) \\ &= \mathcal{F}\left(K_t * (K_s * f)\right) \Longrightarrow K_{t+s} * f = K_t * (K_s * f).\end{aligned}$$

So, in other words, this proves $T(t+s) = T(t)T(s)$. We know already that $\lim_{t\searrow 0} T(t)f = f$, so that we conclude that in fact $t \to T(t)f$ is in $C([0,\infty), L^p(\mathbb{R}^n, \mathbb{R}))$.
Finally, $\|T(t)f\|_p \le \|K_t\|_1 \|f\|_p = \|f\|_p$ implies that $T(t)$ is a contraction semigroup in $L^p(\mathbb{R}^n, \mathbb{R})$.
We know by Proposition 5.8 that $T(t) = e^{tL}$ for L an m–dissipative operator in $L^p(\mathbb{R}^n, \mathbb{R})$. We want to check that $L = \triangle$ with

$$D(\triangle) := \{f \in L^p(\mathbb{R}^n, \mathbb{R}) : \triangle f \in L^p(\mathbb{R}^n, \mathbb{R})\}. \tag{5.14}$$

Notice that we know that this is true in the case $p = 2$.

First of all we observe that $G(\triangle) \supseteq G(L)$. Indeed, in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ we have

$$\mathcal{F}(e^{tL} f) = e^{-t|\xi|^2} \widehat{f}$$

and in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$

$$\mathcal{F}(Lf) = \mathcal{F}\left(\lim_{t\searrow 0} \frac{e^{tL}f - f}{t}\right) = \lim_{t\searrow 0} \mathcal{F}\left(\frac{e^{tL}f - f}{t}\right) = \lim_{t\searrow 0} \frac{e^{-t|\xi|^2} - 1}{t} \widehat{f} = -|\xi|^2 \widehat{f} = \mathcal{F}(\triangle f).$$

So if $f \in D(L)$ we have $Lf = \triangle f$ and so $f \in D(\triangle)$.
Let now $f \in D(\triangle)$. In $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ we have

$$\mathcal{F}\left(\frac{e^{tL}f - f}{t}\right) = \frac{e^{-t|\xi|^2} - 1}{t} \widehat{f} = \frac{\int_0^t e^{-s|\xi|^2} ds}{t} (-|\xi|^2) \widehat{f} = \mathcal{F}\left(t^{-1} \int_0^t e^{sL} \triangle f ds\right).$$

In particular, this implies that in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, but so also in $L^p(\mathbb{R}^n, \mathbb{R})$, we have

$$\frac{e^{tL}f - f}{t} = t^{-1} \int_0^t e^{sL} \triangle f ds.$$

Then

$$Lf = \lim_{t \searrow 0} \frac{e^{tL}f - f}{t} = \lim_{t \searrow 0} t^{-1} \int_0^t e^{sL} \triangle f ds = \triangle f.$$

So $\triangle$ with domain (5.14) is the generator of $T(t)f = K_t * f$ and in particular is m–dissipative.

All the above arguments can be repeated in the context of the space $C_0(\mathbb{R}^n, \mathbb{R})$ based, as they are, on Theorem 1.9.

6 Bochner integral

Let X be a Banach space.

Definition 6.1. Let I be an open interval. A function $f : I \to X$ is measurable if there exists a set E of measure 0 and a sequence $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$.

Lemma 6.2. *Consider the notation of Def. 6.1. Then the function $t \to \|f(t)\|$ is measurable.*

Proof. In the notation of Def. 6.1 the sequence $(\|f_n(t)\|)$ in $C_c(I, \mathbb{R})$ is s.t. $\|f_n(t)\| \to \|f(t)\|$ for any $t \in I \backslash E$. Then $\|f(t)\|$ is measurable, see for example Theorem 1.14 [5]. □

Proposition 6.3. *If (f_n) is a sequence of measurable functions from I to X convergent a.e. to a $f : I \to X$, then f is measurable.*

Proof. There is an E with $|E| = 0$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$. Consider for any n a sequence $C_c(I, X) \ni f_{n,k} \to f_n$ a.e.. We will suppose now that $|I| < \infty$, by the proof can be extended to the case $|I| = \infty$ by expressing $I = \cup_l I_l$ for an increasing sequence of intervals with $|I_l| < \infty$. By applying Egorov Theorem to $\|f_{n,k} - f_n\|$ there is $E_n \subset I$ with $|E_n| \le 2^{-n}$ s.t. $f_{n,k} \to f_n$ uniformly in $I \backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I \backslash E_n$ and set $g_n = f_{n,k(n)}$. Set $F = E \bigcup (\bigcap_m \bigcup_{n>m} E_n)$. Then $|F| = 0$. Indeed for any m

$$|F| \le |E| + \sum_{n=m}^{\infty} |E_n| \overset{m\to\infty}{\to} 0.$$

Let $t \in I \backslash F$. Since $t \notin E$ we have $f_n(t) \to f(t)$. Furthermore, for n large enough we have $t \in I \backslash E_n$. Indeed

$$t \notin \bigcap_m \bigcup_{n>m} E_n \Rightarrow \exists\, m \text{ s.t. } t \notin \bigcup_{n>m} E_n \Rightarrow \ t \notin E_n \ \forall\, n > m.$$

Then $\|g_n(t) - f_n(t)\| < 1/n$ and $g_n(t) \to f(t)$. So $f(t)$ is measurable.

□

Definition 6.4. A measurable function $f : I \to X$ is integrable if there exists a sequence $(f_n(t))$ in $C_c(I, X)$ s.t.

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X dt = 0. \tag{6.1}$$

Notice that $\|f_n(t) - f(t)\|_X$ is measurable by Lemma 6.2.

Proposition 6.5. *Let $f : I \to X$ be integrable. Then there exists an $x \in X$ s.t. if $(f_n(t))$ is a sequence in $C_c(I, X)$ satisfying* (6.1) *then we have*

$$\lim_{n\to\infty} x_n = x \text{ where } x_n := \int_I f_n(t)dt. \tag{6.2}$$

Proof. First of all we check that x_n is Cauchy. This follows immediately from (6.1) and from

$$\begin{aligned}\|x_n - x_m\|_X &= \| \int_I (f_n(t) - f_m(t))dt\|_X \le \int_I \|f_n(t) - f_m(t))\|_X dt \\ &\le \int_I \|f_n(t) - f(t))\|_X dt + \int_I \|f(t) - f_m(t))\|_X dt.\end{aligned}$$

Let us set $x = \lim x_n$. Let $(g_n(t))$ be another sequence in $C_c(I, X)$ satisfying (6.1). Then $\lim \int_I g_n = x$ by

$$\begin{aligned}\| \int_I g_n(t)dt - x\|_X &= \| \int_I (g_n(t) - f_n(t))dt + \int_I f_n(t)dt - x\|_X \\ &\le \int_I \|g_n(t) - f_n(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt \\ &\le \int_I \|g_n(t) - f(t)\|_X dt + \int_I \|f_n(t) - f(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt.\end{aligned}$$

□

Definition 6.6. Let $f : I \to X$ be integrable and let $x \in X$ be the corresponding element obtained from Proposition 6.5. The we set $\int_I f(t)dt = x$.

Theorem 6.7 (Bochner's Theorem). *Let $f : I \to X$ be measurable. Then f is integrable if and only if $\|f\|$ is integrable. Furthermore, we have*

$$\| \int_I f(t)dt\| \le \int_I \|f(t)\|dt. \tag{6.3}$$

Proof. Let f be integrable. Then there is a sequence $(f_n(t))$ in $C_c(I, X)$ satisfying (6.1). We have $\|f\| \le \|f_n\| + \|f - f_n\|$. Since both functions in the r.h.s. are integrable and $\|f\|$ is measurable it follows that$\|f\|$ is integrable.

Conversely let $\|f\|$ be integrable.Then there exists a sequence $(g_n(t))$ in $C_c(I, \mathbb{R})$ s.t. $\int_I |g_n(t) - \|f(t)\||dt \to 0$ and $|g_n(t)| \le g(t)$ a.e. for a $g \in L^1(I)$. In fact it is possible to choose such a sequence so that $\|g_n - g_m\|_{L^1(I)} < 2^{-n}$ for any n and any $m \geq n$. Then if we set

$$S_N(t) := \sum_{n=1}^{N} |g_n(t) - g_{n+1}(t)|$$

we have $\|S_N\|_{L^1(I)} \leq 1$. Since $\{S_N(t)\}_{N\in\mathbb{N}}$ is increasing, the limit $S(x) := \lim_{n\to+\infty} S_n(t)$ remains defined, is finite a.e. and $\|S\|_{L^1(I)} \leq 1$. Then $|g_n(t)| \leq |g_1(t)| + S(t) =: g(t)$ everywhere, where $g \in L^1(I)$.
Let $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ a.e. . Set

$$u_n(t) := \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t).$$

We have

$$\|u_n(t)\| \leq \frac{|g_n(t)|\ \|f_n(t)\|}{\|f_n(t)\| + \frac{1}{n}} \leq |g_n(t)| \leq g(t).$$

We have

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t) = \lim_{n\to\infty} f_n(t) = f(t) \text{ a.e.}.$$

Then we have

$$\lim_{n\to\infty} \|u_n(t) - f(t)\| = 0 \text{ a.e. with } \|u_n(t) - f(t)\| \leq g(t) + \|f(t)\| \in L^1(I)$$

and by dominated convergence we conclude

$$\lim_{n\to\infty} \int_I \|u_n(t) - f(t)\| dt = 0.$$

This implies that f is integrable. Finally, we have

$$\| \int_I f(t) dt\| = \lim_{n\to\infty} \| \int_I u_n(t) dt\| \leq \lim_{n\to\infty} \int_I \|u_n(t)\| dt = \int_I \|f(t)\| dt.$$

□

Corollary 6.8 (Dominated Convergence)**.** *Consider a sequence $(f_n(t))$ of integrable functions $I \to X$, $g : I \to \mathbb{R}$ integrable and let $f : I \to X$. Suppose that*

$$\|f_n(t)\| \leq g(t) \text{ for all } n$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t.$$

Then f is integrable with $\int_I f(t) = \lim_n \int_I f_n(t)$.

Proof. By Dominated Convergence in $L^1(I, \mathbb{R})$ we have $\int_I \|f(t)\| = \lim_n \int_I \|f_n(t)\|$. Also, f is measurable. By Bochner's Theorem f is integrable. By the triangular inequality

$$\limsup_n \| \int_I (f(t) - f_n(t))\| \le \lim_n \int_I \|f(t) - f_n(t)\| = 0$$

where the last inequality follows from $\|f(t) - f_n(t)\| \le \|f(t)\| + g(t)$ and the standard Dominated Convergence. □

Definition 6.9. Let $p \in [1, \infty]$. We denote by $L^p(I, X)$ the set of equivalence classes of measurable functions $f : I \to X$ s.t. $\|f(t)\| \in L^p(I, \mathbb{R})$. We set $\|f\|_{L^p(I,X)} := \|\|f\|\|_{L^p(I,\mathbb{R})}$.

Proposition 6.10. *$(L^p(I, X), \| \|_{L^p})$ is a Banach space. $C_c^\infty(I, X)$ is a dense subspace for $p < \infty$.*

Definition 6.11. We denote by $\mathcal{D}'(I, X)$ the space $\mathcal{L}(\mathcal{D}(I, \mathbb{R}), X)$.

Proposition 6.12. *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}, X)$. Set*

$$T_h f(t) = h^{-1} \int_t^{t+h} f(s) ds \text{ for } t \in \mathbb{R} \text{ and } h \neq 0.$$

Then $T_h f \in L^p(\mathbb{R}, X) \cap C_b^0(\mathbb{R}, X)$, where from now on $C_b^0(\mathbb{R}, X) := L^\infty(\mathbb{R}, X) \cap C^0(\mathbb{R}, X)$, and $T_h f \overset{h\to 0}{\to} f$ in $L^p(\mathbb{R}, X)$ and for almost every t.

Proof. The fact that $T_h f$ belongs to $C_b^0(\mathbb{R}, X)$ is rather immediate. Indeed

$$\|T_h f(t)\| \le h^{-1} \int_t^{t+h} \|f(s)\| ds \le h^{-\frac{1}{p}} \left(\int_{\mathbb{R}} \|f(s)\|^p ds \right)^{\frac{1}{p}} = h^{-\frac{1}{p}} \|f\|_{L^p(\mathbb{R},X)}.$$

On the other hand, for $t < t' < t + h$

$$\|T_h f(t) - T_h f(t')\| \le h^{-1} \left[\int_t^{t'} \|f(s)\| ds + \int_{t+h}^{t'+h} \|f(s)\| ds \right] \overset{t'\to t}{\to} 0$$

and a similar argument on convergence on the left guarantees $T_h f \in C^0(\mathbb{R}, X)$.
Notice that by definition we have $C_c(\mathbb{R}, X)$ dense in $L^p(\mathbb{R}, X)$. Notice also that we have $T_h f = \rho_h * f$ with $\rho_h(t) = h^{-1} \chi_{[0,1]}(h^{-1} t)$. Replacing f with a $g \in C_c(\mathbb{R}, X)$ we have like in Theorem 1.9 that $T_h g \overset{h\to 0}{\to} g$ in $L^p(\mathbb{R}, X)$. By density we have also $T_h f \overset{h\to 0^+}{\to} f$ in $L^p(\mathbb{R}, X)$.

Now we consider the pointwise convergence. Let g_n be a sequence in $C_c(\mathbb{R}, X) \cap L^p(\mathbb{R}, X)$ with $g_n \to f$ in $L^p(\mathbb{R}, X)$. Then $T_h g_n(t) \overset{h\to 0^+}{\to} g_n(t)$ for all $t \in \mathbb{R}$. Furthermore we may assume

$$\lim_{n\to\infty} \|f(t) - g_n(t)\| = 0 \text{ for all } t \notin \Omega_n$$

for a 0 measure set Ω. Furthermore there exists a 0 measure set Ω_n' s.t.

$$\lim_{h\to 0^+} h^{-1} \int_t^{t+h} \|f(s) - g_n(s)\| ds = \|f(t) - g_n(t)\| \text{ for all } t \notin \Omega_n'.$$

Set now for $t \notin \Omega \cup \Omega'$ with $\Omega' = \cup \Omega'_n$

$$T_h f(t) - f(t) = T_h g_n(t) - g_n(t) + T_h(f - g_n)(t) + g_n(t) - f(t).$$

For any $\epsilon > 0$ there is $n(\epsilon)$ s.t. for $n > n(\epsilon)$ we have $\|f(t) - g_n(t)\| < \epsilon$. Furthermore we have

$$\limsup_{h \to 0^+} \|T_h(f - g_n)(t)\| \leq \lim_{h \to 0^+} h^{-1} \int_t^{t+h} \|f(s) - g_n(s)\| ds = \|f(t) - g_n(t)\| < \epsilon.$$

Hence for $t \notin \Omega \cup \Omega'$

$$\limsup_{h \to 0^+} \|T_h f(t) - f(t)\| \leq 2\epsilon.$$

By the arbitrariness of $\epsilon > 0$ it follows

$$\lim_{h \to 0^+} \|T_h f(t) - f(t)\| = 0 \text{ for } t \notin \Omega \cup \Omega'.$$

□

Corollary 6.13. *Let $f \in L^1_{loc}(I, X)$ be such that $f = 0$ in $\mathcal{D}'(I, X)$. Then $f = 0$ a.e.*

Proof. First of all we have $\int_J f dt = 0$ for any $J \subset I$ compact. Indeed, let $(\varphi_n) \in \mathcal{D}(I)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n \to \chi_J$ a.e. Then

$$\int_J f dt = \lim_{n \to +\infty} \int_J \varphi_n f dt = 0$$

where we applied Dominated Convergence for the last equality.
Set now $\overline{f}(t) = f(t)$ in J and $\overline{f}(t) = 0$ outside J. The $\overline{f} \in L^1(\mathbb{R}, X)$ and $T_h \overline{f} = 0$ for all $h > 0$. Then $\overline{f}(t) = 0$ for a.e. t by Prop. 6.12. So $f(t) = 0$ for a.e. $t \in J$ by the previous proposition. This implies $f(t) = 0$ for a.e. $t \in I$. □

Corollary 6.14. *Let $g \in L^1_{loc}(I, X)$, $t_0 \in I$, and $f(t) := \int_{t_0}^t g(s) ds$. Then:*

(1) $f' = g$ in the sense of distributions in $\mathcal{D}'(I, X)$;

(2) f is differentiable in the classical sense a.e. with $f' = g$ a.e.

Proof. It is not restrictive to consider the case $I = \mathbb{R}$ and $g \in L^1(\mathbb{R}, X)$. We have

$$T_h g(t) = h^{-1} \int_t^{t+h} g(s) ds = \frac{\int_{t_0}^{t+h} g(s) ds - \int_{t_0}^t g(s) ds}{h} = \frac{f(t+h) - f(t)}{h}.$$

By Proposition 6.12 $T_h g \overset{h \to 0}{\to} g$ for almost every t. This yields (2).

For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle f', \varphi \rangle_{\mathcal{D}'(\mathbb{R}, X) \mathcal{D}(\mathbb{R})} = -\langle f, \varphi' \rangle_{\mathcal{D}'(\mathbb{R}, X) \mathcal{D}(\mathbb{R})} = -\int_{\mathbb{R}} f(t) \varphi'(t) dt.$$

Furthermore

$$\lim_{h\to 0}\frac{\varphi(t+h)-\varphi(t)}{h}=\varphi'(t) \text{ in } L^\infty(\mathbb{R}).$$

So

$$\langle f',\varphi\rangle_{\mathcal{D}'(\mathbb{R},X)\mathcal{D}(\mathbb{R})}=-\lim_{h\to 0}\int_{\mathbb{R}} f(t)\frac{\varphi(t+h)-\varphi(t)}{h}dt=-\lim_{h\to 0}\int_{\mathbb{R}}\varphi(t)\frac{f(t-h)-f(t)}{h}dt$$
$$=\lim_{h\to 0}\int_{\mathbb{R}}\varphi(t)T_{-h}g(t)dt=\langle g,\varphi\rangle_{\mathcal{D}'(\mathbb{R},X)\mathcal{D}(\mathbb{R})}$$

where in the last equality we used $T_{-h}g\to g$ in $L^1(\mathbb{R},X)$ and so also in $\mathcal{D}'(\mathbb{R},X)$ So $f'=g$ in $\mathcal{D}'(I,X)$.

□

Proposition 6.15. *Let $T\in\mathcal{D}'(I,X)$ s.t. $T'=0$ in $\mathcal{D}'(I,X)$. Then there exists an $x_0\in X$ s.t.*

$$\langle T,\varphi\rangle=x_0\int_I\varphi(t)dt \text{ for any } \varphi\in C_c^\infty(I,\mathbb{R}) \tag{6.4}$$

Proof. Let $\vartheta\in C_c^\infty(I,\mathbb{R})$ with $\int_I\vartheta(t)dt=1$ and set $x_0=\langle T,\vartheta\rangle$. Let $[a,b]\subset I$ be s.t. $[a,b]\supseteq\text{supp}\vartheta$. Then set for any $\varphi\in C_c^\infty(I,\mathbb{R})$

$$\psi(t)=\int_{\inf(I)}^t\left(\varphi(s)-\vartheta(s)\int_I\varphi(\sigma)d\sigma\right)ds.$$

Then $\psi\in C_c^\infty(I,\mathbb{R})$ with

$$\psi'(t)=\varphi(t)-\vartheta(t)\int_I\varphi(\sigma)d\sigma.$$

We have

$$0=\langle T,\psi'\rangle=\langle T,\varphi\rangle-\langle T,\vartheta\rangle\int_I\varphi(\sigma)d\sigma=\langle T,\varphi\rangle-x_0\int_I\varphi(\sigma)d\sigma.$$

This implies that $T=x_0$.

□

Definition 6.16. Let $p\in[1,\infty]$. We denote by $W^{1,p}(I,X)$ the space formed by the $f\in L^p(I,X)$ s.t. $f'\in\mathcal{D}'(I,X)$ is also $f'\in L^p(I,X)$ and we set $\|f\|_{W^{1,p}}=\|f\|_{L^p}+\|f'\|_{L^p}$.

Theorem 6.17. *Let $p\in[1,\infty]$ and $f\in L^p(I,X)$. Then the following properties are equivalent.*

(1) $f\in W^{1,p}(I,X)$.

(2) There exists $g\in L^p(I,X)$ s.t. for a.e. t_0 and t in I we have

$$f(t)=f(t_0)+\int_{t_0}^t g(s)ds. \tag{6.5}$$

(3) f is absolutely continuous, weakly differentiable a.e. with weak derivative $g \in L^p(I,X)$.

Proof. (1)⇒(2). Let $t_0 \in I$ and set

$$w(t) = f(t) - \widetilde{f}(t) \text{ with } \widetilde{f}(t) := f(t_0) + \int_{t_0}^{t} f'(s)ds.$$

$\widetilde{f} \in C^0(I,X)$ satisfies the conclusions of Corollary 6.14. So $w'(t) = f' - \widetilde{f}' = 0$ in $\mathcal{D}'(I,X)$. This implies $w = x_0$ in $\mathcal{D}'(I,X)$, so that we can set $w(t) = x_0$ for all t. Then we can pick $f \in C^0(I,X)$ and we can apply this discussion to this specific function. But then necessarily we must have $x_0 = 0$. This yields (2) for $g = f'$.
(2)⇒(1). We can assume that (6.5) holds everywhere. Then we can apply Corollary 6.14 which tells us that $f' = g$ in the sense of distributions. Hence $f \in W^{1,p}(I,X)$.
(2)⇒(3). We assume, changing $f(t)$ in a 0 measure set, that (6.5) holds for all t. Then by Corollary 6.14 f is differentiable a.e. with $f' = g$ a.e. and its distributional derivative is $g \in L^p(I,X)$. Obviously, we can conclude that f is weakly differentiable a.e. with weak derivative g.
We now show that $f : \mathbb{R} \to \mathbb{C}$ is absolutely continuous, that is for any $\epsilon > 0$ there is $\delta > 0$ such that for any set of disjoint intervals (a_1, b_1), ..., (a_N, b_N)

$$\sum_{j=1}^{N}(b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{N} \|f(b_j) - f(a_j)\| < \epsilon.$$

Indeed for the case $p > 1$ we can use

$$\sum_{j=1}^{N} \|f(b_j) - f(a_j)\| \le \int_{\cup_{j=1}^{N}(a_j,b_j)} \|f'(t)\|dt \le | \cup_{j=1}^{N} (a_j, b_j)|^{\frac{1}{p'}} \|f'\|_{L^p(I,X)} \le \delta^{\frac{1}{p'}} \|f'\|_{L^p(I,X)}.$$

For the case $p = 1$ the result is also true. Notice tha if we set $\mu(E) = \int_E \|f'(t)\|dt$ where $\|f'(t)\| \in L^1(I,\mathbb{R})$ a measure remains defined in $\mathbb{R}$. Such $\mu(E)$ is absolutely continous, that is for any $\epsilon > 0$ there is $\delta > 0$ such that $|E| < \delta \Rightarrow \mu(E) < \epsilon$. This implies that $f(t)$ is AC.
(3)⇒(1). Set

$$\varphi(t) = f(t) - f(t_0) - \int_{t_0}^{t} g(t')dt'$$

and let $x' \in X'$. Denote by $h(t)$ the function $t \to \langle \varphi(t), x' \rangle_{XX'}$. It is absolutely continuous and has a.e. derivative equal to 0. Since $h(t_0) = 0$ it follows that $h(t) \equiv 0$. Since this is true for all $x' \in X'$ it follows that

$$f(t) = f(t_0) + \int_{t_0}^{t} g(t')dt' \text{ for all } t.$$

But now we can apply Corollary 6.14 and conclude that $f' = g$ in $\mathcal{D}'(I,X)$. Since $g \in L^p(I,X)$ we conclude that $f \in W^{1,p}(I,X)$.

□

7 Inhomogeneous equations

Let $T > 0$ and $f : [0, T] \to X$. We consider the problem

$$u \in C^0([0,T], D(A)) \cap C^1([0,T], X) \tag{7.1}$$

$$u'(t) = Au(t) + f(t) \tag{7.2}$$

$$u(0) = x. \tag{7.3}$$

The first step consists in showing that (7.1)–(7.3) can be expressed in integral form. We will later check conditions under which the integral formulation yields (7.1)–(7.3).

Lemma 7.1. *Let $x \in D(A)$ and $f \in C^0([0,T], X)$. Let u be a solution of* (7.1)–(7.3). *Then we have*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds \text{ for all } t \in [0,T]. \tag{7.4}$$

Proof. The elementary way to prove Lemma (7.1) would be the classical *integrating factor* argument for linear first order ODE's, that is apply e^{-tA} to equation (7.2) thus getting

$$(e^{-tA}u(t))' = e^{-tA}f(t)$$

and then, by integration,

$$e^{-tA}u(t) = u(0) + \int_0^t e^{-sA}f(s)ds \Rightarrow u(t) = e^{tA}u(0) + \int_0^t e^{(t-s)A}f(s)ds.$$

The problem with the above formal argument is that e^{-tA} might not exist. So we need a more elaborated version of the integrating factor argument. This goes as follows. For $t \in (0, T]$ and $s \in [0, t]$ we set

$$w(s) := T(t-s)u(s).$$

Notice that since $u \in C^0([0,T], X)$ and $s \to T(t-s)$ is strongly continuous and bounded, then $w(s) \in C^0([0,t], X)$.
For $s \in [0, t)$ and $h \in (0, t-s]$ we write

$$\begin{aligned} &w(s+h) - w(s) = T(t-s-h)u(s+h) - T(t-s)u(s) = \\ &T(t-s-h)(u(s+h) - u(s)) + (T(t-s-h) - T(t-s))u(s) = \\ &T(t-s-h)\left(u(s+h) - u(s) - (T(h) - 1)u(s)\right). \end{aligned}$$

Then and as $h \searrow 0$ we get the right derivative

$$\begin{aligned} &\frac{w(s+h) - w(s)}{h} = \\ &T(t-s-h)\left\{\frac{u(s+h) - u(s)}{h} - \frac{T(h) - 1}{h}u(s)\right\} \to (\frac{d}{ds})_r w(s) = T(t-s)\left\{u'(s) - Au(s)\right\} = T(t-s)f(s). \end{aligned}$$

For $s \in (0,t]$ and $h \in (-s,0)$

$$
\begin{aligned}
&w(s+h) - w(s) = T(t-s-h)u(s+h) - T(t-s)u(s) = \\
&T(t-s)((T(-h)-1)u(s+h) + u(s+h) - u(s)) = \\
&T(t-s)((T(-h)-1)u(s) + u(s+h) - u(s)) + T(t-s)(T(-h)-1)(u(s+h) - u(s)).
\end{aligned}
$$

Then and as $h \nearrow 0$ we get the right derivative

$$
\begin{aligned}
&\frac{w(s+h)-w(s)}{h} = \\
&T(t-s)\left\{\frac{u(s+h)-u(s)}{h} - \frac{T(-h)-1}{h}u(s)\right\} + T(t-s)\frac{T(-h)-1}{h}(u(s+h)-u(s)) \\
&\to (\frac{d}{ds})_l w(s) = T(t-s)\left\{u'(s) - Au(s)\right\} = T(t-s)f(s).
\end{aligned}
$$

Here we used the fact that

$$
\lim_{h\nearrow 0} \frac{T(-h)-1}{h}(u(s+h)-u(s)) = \lim_{h\nearrow 0} h^{-1}\int_0^{-h} T(s')A(u(s+h)-u(s))ds' = 0
$$

by $u \in C^0([0,T], D(A))$.
Since $T(t-s)f(s) \in C^0([0,t], X)$ we have $w \in C^1([0,t), X)$

$$
w'(s) = T(t-s)f(s). \tag{7.5}
$$

Then for $\tau \in (0,t)$

$$
w(\tau) - w(0) = \int_0^\tau w'(s)ds = \int_0^\tau T(t-s)f(s)ds,
$$

where $w(0) = T(t)x$.
By $w(s) \in C^0([0,t], X)$ by taking the limit $\tau \nearrow t$ on both sides we get

$$
w(t) - T(t)x = \int_0^t T(t-s)f(s)ds
$$

where the l.h.s. is $u(t) - T(t)x$. This yields (7.4). □

Now we give conditions under which (7.4) implies (7.1)–(7.3).

Proposition 7.2. *Let $x \in D(A)$ and $f \in C^0([0,T], X)$. Assume one of the following conditions.*

(i) $f \in L^1([0,T], D(A))$.

(ii) $f \in W^{1,1}([0,T], X)$.

Then u given by (7.4) *is the solution to* (7.1)–(7.3).

Proof. We set for $t \in [0,T]$

$$v(t) := \int_0^t T(t-s)f(s)ds = \int_0^t T(s)f(t-s)ds.$$

Step 1. We prove now that $v \in C^1([0,T),X)$. If (i) holds for $t \in [0,T)$ and $h \in (0,T-t]$ we have

$$\begin{aligned}\frac{v(t+h)-v(t)}{h} &= \frac{\int_0^{t+h} T(t+h-s)f(s)ds - \int_0^t T(t-s)f(s)ds}{h}\\ &= \frac{\int_0^t (T(t+h-s)-T(t-s))f(s)ds + \int_t^{t+h} T(t+h-s)f(s)ds}{h}\\ \frac{v(t+h)-v(t)}{h} &= \int_0^t T(t-s)\frac{T(h)-1}{h}f(s)ds + \frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds. \qquad (7.6)\end{aligned}$$

We take the limit for $h \searrow 0$ and claim that $f \in L^1([0,T],D(A))$ implies

$$\frac{T(h)-1}{h}f(s) = h^{-1}\int_0^h T(\tau)Af(s)d\tau \overset{h\to 0^+}{\to} Af(s) \text{ in } L^1([0,T],X). \qquad (7.7)$$

To prove this notice that $f \in L^1([0,T],D(A))$ implies that there exists a sequence formed by $f_n \in C^0([0,T],D(A))$ s.t. we have $f_n \overset{n\to\infty}{\to} f$ in $L^1([0,T],D(A))$. We have

$$\begin{aligned}&h^{-1}\int_0^h T(\tau)Af(s)d\tau - Af(s) = h^{-1}\int_0^h (T(\tau)-1)Af_n(s)d\tau\\ &+h^{-1}\int_0^h T(\tau)A(f(s)-f_n(s))d\tau + A(f_n(s)-f(s)).\end{aligned}$$

Notice that for any $\epsilon > 0$ there exists n_0 s.t. $n \geq n_0$ and $h \in (0,T]$ imply that the last line has $L^1([0,T],X)$ norm less than ϵ. On the other hand for any n and for $h \searrow 0$ the first term on the r.h.s. converges to 0 in $L^1([0,T],X)$. So the limit claimed in (7.7) is proved. Similarly we have

$$\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds \overset{h\to 0^+}{\to} f(t) \qquad (7.8)$$

which follows from $f \in C^0([0,T],X)$. Then taking the limit in (7.6) we have

$$\frac{d^+}{dt}v(t) = \int_0^t T(t-s)Af(s)ds + f(t). \qquad (7.9)$$

Hence we have proved under hypothesis (i) that $\frac{d^+}{dt}v(t) \in C^0([0,T),X)$.

Assume now case (ii).

$$\begin{aligned}&\frac{v(t+h)-v(t)}{h} = \frac{\int_0^{t+h} T(s)f(t+h-s)ds - \int_0^t T(s)f(t-s)ds}{h} =\\ &\frac{\int_0^t T(s)(f(t+h-s)-f(t-s))ds + \int_t^{t+h} T(s)f(t+h-s)ds}{h} =\\ &\int_0^t T(s)\frac{f(t+h-s)-f(t-s)}{h}ds + \frac{1}{h}T(h)\int_0^h T(t-s)f(s)ds.\end{aligned}$$

We have

$$\lim_{h\searrow 0} \frac{f(t+h-s)-f(t-s)}{h} = f'(t-s) \text{ in } L^1([0,t),X).$$

Then we have

$$\frac{d^+}{dt}v(t) = \int_0^t T(s)f'(t-s)ds + T(t)f(0) \tag{7.10}$$

Hence also under hypothesis (ii) we have proved that $\frac{d^+}{dt}v(t) \in C^0([0,T),X)$.

Step 2. By similar arguments $\frac{d^-}{dt}v(t) \in C^0((0,T],X)$.

For example, if (i) holds for $t \in (0,T]$ and $h > 0$ is small we have

$$\begin{aligned}
\frac{v(t-h)-v(t)}{-h} &= \frac{\int_0^{t-h} T(t-h-s)f(s)ds - \int_0^t T(t-s)f(s)ds}{-h} \\
&= \frac{\int_0^{t-h} (T(t-h-s)-T(t-s))f(s)ds - \int_{t-h}^t T(t-s)f(s)ds}{-h} \\
\frac{v(t+h)-v(t)}{h} &= \int_0^t T(t-h-s)\frac{1-T(h)}{-h}f(s)ds + \frac{1}{h}\int_{t-h}^t T(t-s)f(s)ds.
\end{aligned}$$

As $h \searrow 0$ the limit (7.7) holds and the above converges to

$$\frac{d^-}{dt}v(t) = \int_0^t T(t-s)Af(s)ds + f(t).$$

Notice that for $t \in (0,T)$ we have $\frac{d^+}{dt}v(t) = \frac{d^-}{dt}v(t)$.

Step 3. Let $t \in [0,T)$ and $h \in [0,T-t)$. Then

$$\begin{aligned}
&\frac{T(h)-1}{h}v(t) = \frac{T(h)-1}{h}\int_0^t T(t-s)f(s)ds \\
&= h^{-1}\int_0^t T(t+h-s)f(s)ds - h^{-1}\int_0^t T(t-s)f(s)ds \\
&= \underbrace{h^{-1}\int_0^{t+h} T(t+h-s)f(s)ds - h^{-1}\int_0^t T(t-s)f(s)ds}_{\frac{v(t+h)-v(t)}{h}} - h^{-1}\int_t^{t+h} T(t+h-s)f(s)ds.
\end{aligned} \tag{7.11}$$

So

$$\frac{T(h)-1}{h}v(t) \overset{h\to 0^+}{\to} \frac{d^+}{dt}v(t) - f(t). \tag{7.12}$$

Then $v(t) \in D(A)$ with

$$Av(t) = v'(t) - f(t) \text{ for } t \in [0,T). \tag{7.13}$$

Notice that by Step 1 and 2 we have $v' \in C([0,T],X)$. So since $G(A)$ is closed we conclude that also $v(T) \in D(A)$.

We now discuss the fact that $v \in C^0([0,T], D(A))$. Here we know already that $v \in C^0([0,T], X)$ and what remains to be shown is $Av \in C([0,T], X)$. If $f \in W^{1,1}([0,T], X)$ this follows immediately from (7.13) (that holds also at T).
If $f \in L^1([0,T], D(A))$ then

$$Av(t) = A\int_0^t T(t-s)f(s)ds = \int_0^t T(t-s)Af(s)ds \tag{7.14}$$

where we claim that the last term in $C^0([0,T], X)$. To prove this claim, let (f_n) in $C^0([0,T], D(A))$ with $f_n \to f$ in $L^1([0,T], D(A))$. Then

$$Av(t) = \underbrace{\int_0^t T(t-s)Af_n(s)ds}_{=:\varphi_n(t)} + \int_0^t T(t-s)(Af(s) - Af_n(s))ds$$

Then we see that

$$\|Av(t) - \varphi_n(t)\|_X \le \|f - f_n\|_{L^1([0,T],D(A))}.$$

This implies $\varphi_n \to Av$ in $L^\infty([0,T], X)$ and this, in turn, implies $Av \in C^0([0,T], X)$.

Step 4. Since u satisfies (7.4) we have $u(t) = T(t)x + v(t)$. The r.h.s. is in $C^0([0,T], D(A)) \cap C^1([0,T], X)$. This yields (7.1). We have $u'(t) = AT(t)x + Av(t) + f(t) = Au(t) + f(t)$ for all $t \in [0,T]$. So (7.2) holds. Finally $u(0) = x$ follows.

□

Corollary 7.3. *Let $x \in X$ and $f \in C^0([0,T], X)$ and let u be given by (7.4). Then we have*

$$u \in C^0([0,T], X) \cap C^1([0,T], \overline{X}) \tag{7.15}$$

$$u'(t) = \overline{A}u(t) + f(t) \tag{7.16}$$

$$u(0) = x. \tag{7.17}$$

Proof. Recall that $X = D(\overline{A})$. We have $f \in C^0([0,T], D(\overline{A})) \subseteq L^1([0,T], D(\overline{A})) \cap C^0([0,T], \overline{X})$ and $x \in D(\overline{A})$. So we can apply the Proposition 7.2.

□

Corollary 7.4. *Let $x \in X$ and $f \in C^0([0,T], X)$ and let u be given by (7.4). Assume u satisfies one of the following 2 conditions.*

(i) $u \in C^0([0,T], D(A))$.

(ii) $u \in C^1([0,T], X)$.

Then u satisfies (7.1)–(7.3).

Proof. In case (i) we can apply the previous corollary. In particular we get (7.16). But (i) implies $\overline{A}u(t) = Au(t)$. So (i) implies $u'(t) = Au(t) + f(t)$. Since the right hand side is in $C^1([0,T], X)$ than this and (i) imply $u \in C^1([0,T], X)$. So we get (7.1)–(7.3).

Now assume case (ii). Solving (7.16) with respect to $\overline{A}u(t)$ we see that $\overline{A}u \in C^0([0,T], X)$. But this implies $\overline{A}u(t) = Au(t)$ and $u \in C^0([0,T], D(A))$. But then get (7.1)–(7.3).

□

Proposition 7.5. *Let $x \in X$, $f, u \in L^1([0,T], X)$. Assume that either $u \in L^1([0,T], D(A))$ or $u \in W^{1,1}([0,T], X)$. Then u satisfies (7.4) if and only if*

$$u \in L^1([0,T], D(A)) \cap W^{1,1}([0,T], X) \tag{7.18}$$

$$u'(t) = Au(t) + f(t) \text{ for almost any } t \in [0,T] \tag{7.19}$$

$$u(0) = x. \tag{7.20}$$

Proof. If $u \in W^{1,1}([0,T], X)$ then $u \in C^0([0,T], X)$ and so (7.20) makes sense. We now show that (7.18)–(7.20) imply (7.4). We proceed like in Lemma 7.1. For $t \in (0,T]$ and $s \in (0,t)$ we set

$$w(s) := T(t-s)u(s).$$

For $h \in (0, t-s)$ we have already computed that

$$\frac{w(s+h) - w(s)}{h} = T(t-s-h)\left\{\frac{u(s+h) - u(s)}{h} - \frac{T(h) - 1}{h}u(s)\right\} \tag{7.21}$$

and so

$$w(s+h) - w(s) = T(t-s-h)\left\{u(s+h) - u(s) - (T(h) - 1)u(s)\right\} \tag{7.22}$$

From this we see

$$\begin{aligned}\|w(s+h) - w(s)\|_X &\leq \|u(s+h) - u(s)\|_X + |h| \; \|u(s)\|_{D(A)} \\ &\leq \|u'\|_{L^1((s,s+h),X)} + |h| \; \|u(s)\|_{D(A)}.\end{aligned}$$

This implies that w is absolutely continuous from $[0,T]$ to X.
The fact that $u \in L^1([0,T], D(A)) \cap W^{1,1}([0,T], X)$ implies that for a.e. s the limit for $h \searrow$ in (7.21) exists with

$$\frac{d^+}{dt}w(s) = T(t-s)(u'(s) - Au(s)) = T(t-s)f(s). \tag{7.23}$$

Similarly we have

$$\frac{d^-}{dt}w(s) = T(t-s)f(s) \text{ a.e.} \tag{7.24}$$

and so

$$\frac{d}{dt}w(s) = T(t-s)f(s) \text{ a.e.}. \tag{7.25}$$

So now we have $w \in AC([0,T], X)$, the derivative w' defined a.e. and is a function belonging to $L^1([0,T], X)$. Notice that w satisfies the hypotheses of claim (3) in Theorem 6.17. This claim guarantees that under these hypotheses $w \in W^{1,1}([0,T], X)$. We have

$$u(t) = w(t) = w(0) + \int_0^t T(t-s)f(s)ds = T(t)x + \int_0^t T(t-s)f(s)ds. \tag{7.26}$$

that is (7.4).

We prove now that (7.4) implies (7.18)–(7.20).

Let (f_n) a sequence in $C^0([0,T],X)$ s.t. $f_n \to f$ in $L^1([0,T],X)$ and let (u_n) the corresponding sequence of solutions of (7.4). Notice that for each n we are under the hypotheses of Corollary 7.3. So we have

$$\begin{aligned}&u_n \in C^0([0,T],X) \cap C^1([0,T],\overline{X})\\&u_n'(t) = \overline{A}u_n(t) + f_n(t) \text{ in } [0,T]\\&u_n(0) = x.\end{aligned}$$

In particular we conclude

$$u_n(t) = x + \int_0^t (\overline{A}u_n(s) + f_n(s))ds \text{ for all } t \in [0,T]. \tag{7.27}$$

Notice that we have also

$$u_n(t) = T(t)x + \int_0^t e^{(t-s)\overline{A}} f_n(s)ds = T(t)x + \int_0^t e^{(t-s)A} f_n(s)ds \text{ for all } t \in [0,T].$$

Then for $n \to \infty$ and by (7.4)

$$\lim_{n\to\infty} u_n(t) = T(t)x + \int_0^t e^{(t-s)A} f(s)ds = u(t) \text{ for all } t \in [0,T] \text{ and in } X.$$

More precisely, we have

$$\|u(t) - u_n(t)\|_X \le \|f - f_n\|_{L^1([0,T],X)}.$$

This implies that $u_n(t) \to u(t)$ in $C^0([0,T],X)$ or, equivalently, in $C^0([0,T],D(\overline{A}))$. Then for $n \to +\infty$ we obtain

$$u(t) = x + \int_0^t (\overline{A}u(s) + f(s))ds \text{ for all } t \in [0,T]. \tag{7.28}$$

It follows that $u \in W^{1,1}([0,T],\overline{X})$ with $u'(t) = \overline{A}u(t) + f(t)$ for almost any t. Since either $u \in L^1([0,T],D(A))$ or $u \in W^{1,1}([0,T],X)$ and since $f \in L^1([0,T],X)$ we have in fact (7.19).

If $u \in W^{1,1}([0,T],X)$ by hypothesis, then from (7.19) and $f \in L^1([0,T],X)$ we get $Au \in L^1([0,T],X)$. This implies $u \in L^1([0,T],D(A))$ and proves (7.18).

If $u \in L^1([0,T],D(A))$ by hypothesis, then from (7.19) and $f \in L^1([0,T],X)$ we get $u' \in L^1([0,T],X)$. This implies $u \in W^{1,1}([0,T],X)$ and proves (7.18). □

Lemma 7.6 (Gronwall's inequality)**.** *Let $T > 0$, $\lambda, \varphi \in L^1(0,T)$ both ≥ 0 a.e. and C_1, C_2 both ≥ 0. Let $\lambda\varphi \in L^1(0,T)$ and let*

$$\varphi(t) \le C_1 + C_2 \int_0^t \lambda(s)\ \varphi(s)ds \text{ for a.e. } t \in (0,T).$$

Then we have

$$\varphi(t) \le C_1 e^{C_2 \int_0^t \lambda(s)ds} \text{ for a.e. } t \in (0,T).$$

Proof. Set

$$\psi(t) := C_1 + C_2 \int_0^t \lambda(s)\ \varphi(s) ds.$$

Then $\psi(t)$ is absolutely continuous and so it is differentiable almost everywhere and we have

$$\psi'(t) = C_2 \lambda(t)\ \varphi(t) \le C_2 \lambda(t)\ \psi(t) \text{ for a.e. } t \in (0, T).$$

Also, the function $\psi(t) e^{-C_2 \int_0^t \lambda(s) ds}$ is absolutely continuous with

$$\frac{d}{dt} \left(\psi(t) e^{-C_2 \int_0^t \lambda(s) ds} \right) \le 0 \text{ for a.e. } t \in (0, T).$$

Then we have

$$\psi(t) \le e^{C_2 \int_0^t \lambda(s) ds} \psi(0) = C_1 e^{C_2 \int_0^t \lambda(s) ds} \text{ for all } t \in (0, T).$$

Since $\varphi(t) \le \psi(t)$ a.e., the result follows.

8 Abstract semilinear equations

Definition 8.1. A function $F : X \to X$ is Lipschitz continuous on bounded subsets of X if for any $M > 0\ \exists\ L(M)$ s.t.

$$\|F(x) - F(y)\| \le L(M) \|x - y\| \text{ for all } x, y \text{ with } \|x\| \le M \text{ and } \|y\| \le M. \tag{8.1}$$

Lemma 8.2. *Let $T > 0$, $x \in X$ and let $u, v \in C^0([0, T], X)$ solve*

$$w(t) = T(t)x + \int_0^t T(t-s) F(w(s)) ds. \tag{8.2}$$

Then $u = v$.

Let $M = \max_{0 \le t \le T} \{\|u(t)\|, \|v(t)\|\}$. Then

$$\|u(t) - v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\| ds \le L(M) \int_0^t \|u(s) - v(s)\| ds$$

and apply Gronwall's inequality. □

Proposition 8.3. *Let $x \in X$ with $\|x\| \le M$. Then there is a unique solution $u \in C^0([0, T_M], X)$ of* (8.2) *with*

$$T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}. \tag{8.3}$$

Proof. Set $K = 2M + \|F(0)\|$ and

$$E = \{u \in C^0([0,T_M],X) : \|u(t)\| \le K \text{ for all } t \in [0,T_M]\}$$

with the distance of $L^\infty([0,T_M],X)$. E is a complete metric space. Next consider the map $u \in E \to \Phi_u$

$$\Phi_u(t) = T(t)x + \int_0^t T(t-s)F(u(s))ds \text{ for all } t \in [0,T_M].$$

By $T_M = \frac{1}{2(L(K)+1)}$ for all $t \in [0,T_M]$ we have

$$\begin{aligned} &\|F(u(t))\| \le \|F(0)\| + \|F(u(t)) - F(0)\| \le \|F(0)\| + KL(K) \\ &= \|F(0)\| + (2M + \|F(0)\|)L(K) \le 2(M + \|F(0)\|)(L(K)+1) = \frac{M + \|F(0)\|}{T_M} \end{aligned} \tag{8.4}$$

and

$$\|T(t)x\| \le \|x\| \le M. \tag{8.5}$$

So from (8.4)–(8.5) for $t \in [0,T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M + \|F(0)\|}{T_M} \le 2M + \|F(0)\| = K$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le T_M L(K)\|u - v\|_{L^\infty([0,T],X)}.$$

So by $T_M L(K) < 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T],X)} \le 2^{-1}\|u - v\|_{L^\infty([0,T],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point.

□

Notice that if $F(0) = 0$ if and $\lim_{M \to 0^+} L(M) = 0$, something which happens in many important cases, we can improve the above result and get a T_M s.t. $\lim_{M \to 0^+} T_M = \infty$, as we will see now.

Proposition 8.4. *Let $x \in X$ with $\|x\| \le M$. Assume $F(0) = 0$ Then there is a unique solution $u \in C^0([0,T_M],X)$ of (8.2) with*

$$T_M := \frac{1}{2L(2M)}. \tag{8.6}$$

Proof. The argument is the same. Here we set $K = 2M$ and define E as above by

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le 2M \text{ for all } t \in [0, T_M]\}$$

Consider the map $u \in E \to \Phi_u$ defined as above by

$$\Phi_u(t) = T(t)x + \int_0^t T(t-s)F(u(s))ds \text{ for all } t \in [0, T_M].$$

By $T_M = \frac{1}{2L(2M)}$ for all $t \in [0, T_M]$ we have

$$\|F(u(t))\| \le 2ML(2M) = \frac{M}{T_M} \tag{8.7}$$

and

$$\|T(t)x\| \le \|x\| \le M. \tag{8.8}$$

So from (8.4)–(8.5) for $t \in [0, T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M}{T_M} \le 2M$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\| ds \le T_M L(2M) \|u - v\|_{L^\infty([0,T],X)}.$$

So by $T_M L(2M) = 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T],X)} \le 2^{-1} \|u - v\|_{L^\infty([0,T],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point. □

We now turn to an abstract form of the *maximum principle.*

Recall that in an ordered Banach space the ordering is characterized by a convex closed cone $\mathcal{C}$ s.t.

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$
2. $\lambda\mathcal{C} \subseteq \mathcal{C}$ for all $\lambda \ge 0$ and
3. $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

Then given $x, y \in X$ we write $y \ge x$ if $(y - x) \in \mathcal{C}$.

Lemma 8.5. *Suppose that in X there is a relation of order and that $F(u) \ge 0$ if $u \ge 0$. Suppose furthermore that $T(t)$ is positivity preserving, that is $x \ge 0 \Rightarrow T(t)x \ge 0$ for all t. Then if $x \ge 0$ the solution $u \in C^0([0, T_M], X)$ of Prop. 8.3 is $u(t) \ge 0$ for all t.*

Proof. We just rephrase the fixed point argument of Prop. 8.3 in a different set up. Indeed, if we redefine the set E writing

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le K \text{ and } u(t) \ge 0 \text{ for all } t \in [0, T_M]\},$$

then E is a complete metric space. Furthermore the map $u \to \Phi_u$ with

$$\Phi_u(t) = T(t)f + \int_0^t T(t-s)F(u(s))ds \text{ for all } t \in [0, T_M].$$

is such that $u(t) \ge 0$ for all $t \in [0, T_M]$ implies $\Phi_u(t) \ge 0$ for all $t \in [0, T_M]$. Then the proof of Proposition 8.3 works out in the same way as before under this slightly more restrictive definition of E.

□

Lemma 8.6. *Assume the hypotheses of Lemma 8.5 and furthermore that $F(v) \ge F(u) \ge 0$ if $v \ge u \ge 0$. Let $x < y$. Let $u(t), v(t) \in C^0([0, T_*), X)$ be solutions with $u(0) = x$ and $v(0) = y$. Then $u(t) \le v(t)$ in $[0, T_*)$.*

Proof. If $M = \max\{\|x\|, \|y\|\}$, then using the setup of Prop. 8.3 we consider the set

$$E = \{f \in C^0([0, T_M], X) : f(t) \ge 0 \text{ and } \|f(t)\| \le K \text{ for all } t \in [0, T_M]\}$$

and the maps $f \in E \to \Phi_x(f)$ and $f \in E \to \Phi_y(f)$

$$\Phi_w(f)(t) = T(t)w_0 + \int_0^t T(t-s)F(f(s))ds \text{ for all } t \in [0, T_M].$$

Let $u(t)$ be the solution with initial datum y. Then we have $\Phi_x(u) < \Phi_y(u) = u$. This can be iterated and if $0 < \Phi_x^j(u) < \Phi_x^{j-1}(u)$, then $0 < \Phi_x^{j+1}(u) < \Phi_x^j(u)$. But we know that $\Phi_x^j(u) \overset{j\to\infty}{\to} v$, with v the solution with initial datum x. Hence $v \le u$.

So we have proved $u(t) \le v(t)$ in $[0, T_M]$. Let now

$$T_1 := \sup\{T \in [0, T_*) \text{ such that } u(t) \le v(t) \text{ in } [0, T]\}.$$

If $T_1 = T_*$ the theorem is finished. If $T_1 < T_*$ we have by continuity $u(T_1) \le v(T_1)$. But then there exists a $0 < T < T_* - T_1$ with s.t. $\widetilde{u}(t) := u(t + T_1)$ and resp. $\widetilde{v}(t) := v(t + T_1)$ solve in $[0, T]$ the equation with initial data $\widetilde{x} \le \widetilde{y}$ with $\widetilde{x} := u(T_1)$ and resp. $\widetilde{y} := v(T_1)$. But for T small enough we have $\widetilde{u}(t) \le \widetilde{v}(t)$ in $[0, T]$ by the 1st part of the proof. But this implies than $u(t) \le v(t)$ in $[0, T_1 + T]$. This is absurd by the definition of T_1, and so $T_1 = T_*$.

□

We will consider now the function $T : X \to (0, \infty]$ where for any $x \in X$ the interval $[0, T(x))$ is the maximal (positive) interval of existence of the unique solution of (8.2).

Theorem 8.7. *We have, for $u(t)$ the corresponding solution in $C([0,T(x)),X)$,*

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x)-t} - 2 \tag{8.9}$$

for all $t \in [0,T(x))$. We have the alternatives

(1) $T(x) = +\infty$;

(2) if $T(x) < +\infty$ then $\lim_{t \nearrow T(x)} \|u(t)\| = +\infty$.

Proof. First of all it is obvious that if $T(x) < +\infty$ then by (8.9)

$$\lim_{t\nearrow T(x)} L(\|F(0)\| + 2\|u(t)\|) = +\infty \Rightarrow \lim_{t\nearrow T(x)} \|u(t)\| = +\infty$$

where the implication follows from the fact that $M \to L(M)$ is an increasing function.

Let $x \in X$. Set $T(x) = \sup\{T > 0 : \exists u \in C^0([0,T),X) \text{ solution of (8.2) }\}$. We are left with the proof of (8.9), which is clearly true if $T(x) = \infty$. Now suppose that $T(x) < \infty$ and that (8.9) is false. This means that there exists a $t \in [0,T(x))$ with

$$\frac{1}{T_M} - 2 = 2L(\|F(0)\| + 2\|u(t)\|) < \frac{1}{T(x)-t} - 2 \Rightarrow T(x) - t < T_M$$

for $M = \|u(t)\|$, where we recall $T_M := \frac{1}{2L(2M+\|F(0)\|)+2}$ in (8.3). Consider now $v \in C^0([0,T_M],X)$ the solution of

$$v(s) = T(s)u(t) + \int_0^s T(s-s')F(v(s'))ds' \text{ for all } s \in [0,T_M].$$

which exists by the previous Proposition 8.3. Then define

$$w(s) := \begin{cases} u(s) \text{ for } s \in [0,t] \\ v(s-t) \text{ for } s \in [t,t+T_M]. \end{cases}$$

We claim that $w \in C^0([0,t+T_M],X)$ is a solution of (8.2). In $[0,t]$ this is obvious since in $w = u$ in $[0,t]$ and $u \in C^0([0,t],X)$ is a solution of (8.2). Let now $s \in (t,t+T_M]$. We have

$$\begin{aligned} w(s) &= v(s-t) = T(s-t)u(t) + \int_0^{s-t} T(s-t-s')F(v(s'))ds' \\ &= T(s-t)\left[T(t)x + \int_0^t T(t-s')F(u(s'))ds'\right] + \int_0^{s-t} T(s-t-s')F(v(s'))ds' \\ &= T(s)x + \int_0^t T(s-s')F(\underbrace{u(s')}_{w(s')})ds' + \int_t^s T(s-s')F(\underbrace{v(s'-t)}_{w(s')})ds' \\ &= T(s)x + \int_0^s T(s-s')F(w(s'))ds. \end{aligned}$$

□

Remark 8.8. Notice that if $F(0) = 0$, then we can prove the improved estimate

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x) - t}. \tag{8.10}$$

The proof is exactly the same of Theorem 8.7 using the altered definitions of T_M, $T_M = (2L(2M))^{-1}$.

Proposition 8.9. *(1) $T : X \to (0, \infty]$ is lower semicontinuous;*

(2) if $x_n \to x$ in X and if $T < T(x)$ the $u_n \to u$ in $C^0([0,T], X)$ with u_n the solution of (8.2) with initial datum x_n.

Proof. Let $u \in C^0([0, T(x)), X)$ the solution of (8.2) and consider $T < T(x)$. Set $M = 2\|u\|_{L^\infty([0,T],X)}$ and let

$$\tau_n = \sup\{t \in [0, T(x_n)) : \|u_n\|_{L^\infty([0,t],X)} \leq K\} \text{ where } K = 2M + \|F(0)\|.$$

For $n \gg 1$ we have $\|x_n\| < M$. Then $u_n \in C^0([0, T_M], X)$ with $\|u_n\|_{L^\infty([0,T_M],X)} \leq K$ by Prop. 8.3. This implies $\tau_n \geq T_M$. For $0 \leq t \leq \min\{T, \tau_n\}$ we have

$$u(t) - u_n(t) = T(t)(x - x_n) + \int_0^t T(s-t)(F(u(s)) - F(u_n(s)))ds$$

and so

$$\begin{aligned} &\|u(t) - u_n(t)\| \leq \|x - x_n\| + L(K) \int_0^t \|u(s) - u_n(s)\| ds \Rightarrow \\ &\|u(t) - u_n(t)\| \leq e^{L(K)t}\|x - x_n\| \Rightarrow \|u(t) - u_n(t)\| \leq e^{L(K)T}\|x - x_n\|. \end{aligned} \tag{8.11}$$

So $\|u_n(t)\| \leq \|u(t)\| + e^{L(K)T}\|x - x_n\| \leq M/2 + e^{L(K)T}\|x - x_n\| \leq M$ for $n \gg 1$ and $0 \leq t \leq \min\{T, \tau_n\}$. This and continuity imply $\tau_n > \min\{T, \tau_n\}$ and so $\tau_n > T$. Then we have $T(x_n) > T$. This implies the lower semi–continuity in claim (1). Furthermore by (8.11) we have also $u_n \to u$ in $C^0([0,T], X)$. □

9 Nonlinear heat equation

We set $X = C_0(\mathbb{R}^n, \mathbb{R})$ and consider a locally Lipschitz map $g \in C^0(\mathbb{R}, \mathbb{R})$. We set $F(u)(x) := g(u(x))$. Recall that X is a closed subspace of $L^\infty(\mathbb{R}^n, \mathbb{R})$. Consider, the operator $\triangle$ with

$$D(\triangle) := \{f \in C_0(\mathbb{R}^n, \mathbb{R}) : \triangle f \in C_0(\mathbb{R}^n, \mathbb{R})\}.$$

We know from Sect. 5.2 that this operator is m–dissipative with corresponding semigroup $e^{t\triangle} f = K_t * f$. Furthermore the functional

$$F : C_0(\mathbb{R}^n, \mathbb{R}) \to C_0(\mathbb{R}^n, \mathbb{R})$$

is locally Lipschitz. We can then apply all the results of Section 8 to the equation.

$$u(t) = e^{t\triangle} f + \int_0^t e^{(t-s)\triangle} F(u(s))ds,$$

which is a formulation of the problem

$$\begin{cases} u_t = \triangle u + F(u) \text{ for } (t,x) \in [0,\infty) \times \mathbb{R}^n \\ u(0,x) = f(x) \end{cases} \tag{9.1}$$

Typical cases can be $F(u) = \lambda |u|^{p-1} u$ for $\lambda \in \mathbb{R}\backslash\{0\}$ and $p > 1$.

9.1 The blowup theorem by Hiroshi Fujita

We consider now the Cauchy problem for the heat equation

$$\begin{cases} u_t = \triangle u + |u|^{p-1} u \text{ with } (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) \text{ where } u_0 \in C_0(\mathbb{R}^n, \mathbb{R}). \end{cases}$$

We first observe that by applying the theory in Section 8 we can prove the following maximum principle property.

Lemma 9.1. *Let $u \in C([0,T), C_0(\mathbb{R}^n, \mathbb{R}))$ be the unique maximal solution of*

$$u(t) = e^{t\triangle} f + \int_0^t e^{(t-s)\triangle} |u(s)|^{p-1} u(s) ds \tag{9.2}$$

and let $f \geq 0$. Then $u(t,x) \geq 0$ for all $(t,x) \in [0,T) \times \mathbb{R}^n$.

□

We now focus on positive solutions of

$$\begin{cases} u_t = \triangle u + |u|^{p-1} u \text{ with } (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) \text{ where } u_0 \in C_0(\mathbb{R}^n, \mathbb{R}) \end{cases} \tag{9.3}$$

Theorem 9.2. *Let $u_0 \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ with $u_0 \geq 0$ and $u_0 \neq 0$ and suppose $1 < p \leq 1 + \frac{2}{n}$. Suppose that $u(t) \in C^0([0, T_{u_0}), C_0(\mathbb{R}^n))$ is a positive solution of*

$$u(t) = e^{t\triangle} u_0 + \int_0^t e^{(t-s)\triangle} u^p(s) ds. \tag{9.4}$$

Then $T_{u_0} < \infty$.

Remark 9.3. The original paper by Fujita [3] deals with the case $1 < p < 1 + \frac{2}{n}$. The proof we give is due to Weissler [7].

Proof. We claim, and for the moment assume, the following inequality due to Weissler:

$t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) \leq C$ for a fixed $C = C(p) > 0$, for any $x \in \mathbb{R}^n$, any $u_0 \geq 0$ and any $t \in [0, T_{u_0})$. (9.5)

Here, crucially, C depends only on p.
Suppose we have $T_{u_0} = \infty$ and assume (9.5).
By dominated convergence we have for any $x \in \mathbb{R}^n$

$$\lim_{t \nearrow \infty} (4\pi)^{\frac{n}{2}} t^{\frac{n}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy = \int_{\mathbb{R}^n} u_0(y) dy = \|u_0\|_{L^1(\mathbb{R}^n)}. \tag{9.6}$$

In the particular case $p < 1 + \frac{2}{n}$, equivalent to $\frac{1}{p-1} - \frac{n}{2} > 0$, we see immediately that (9.6) is incompatible with (9.5) since

$$\lim_{t \nearrow \infty} t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{n}{2}} t^{\frac{n}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{n}{2}} (4\pi)^{-\frac{n}{2}} \|u_0\|_{L^1(\mathbb{R}^n)} = +\infty.$$

In the case $p = 1 + \frac{2}{n}$ this argument does not provide a contradiction for all u_0 (although this argument shows that if $\|u_0\|_{L^1(\mathbb{R}^n)} > (4\pi)^{\frac{n}{2}} C$ for $C = C(1 + \frac{2}{n})$ then there is blow up). We complete the argument below, but first we prove claim (9.5).

Proof of (9.5) We turn now to the proof of (9.5). We have $u(t) \geq e^{t\triangle} u_0(x)$ and

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\triangle} u^p(s) ds \geq \int_0^t e^{(t-s)\triangle} (e^{s\triangle} u_0)^p ds \\ &\geq \int_0^t (e^{(t-s)\triangle} e^{s\triangle} u_0)^p ds = \int_0^t (e^{t\triangle} u_0)^p ds = t (e^{t\triangle} u_0)^p, \end{aligned} \tag{9.7}$$

where we used, for $d\mu(y) := (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4\tau}} dy$ which gives a probability measure in $\mathbb{R}^n$,

$$\begin{aligned} e^{\tau\triangle} (f)^p(x) &= (4\pi\tau)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\tau}} f^p(y) dy = \int_{\mathbb{R}^n} f^p(y) d\mu(y) \\ &\geq \left(\int_{\mathbb{R}^n} f(y) d\mu(y) \right)^p = \left((4\pi\tau)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\tau}} f(y) dy \right)^p = \left(e^{\tau\triangle}(f)(x) \right)^p, \end{aligned}$$

which follows from Jensen's inequality $\varphi(\int f d\mu) \leq \int \varphi \circ f d\mu$ for a convex function φ and a probability measure μ.

By a substitution inside (9.7) and by repeating the same argument we get

$$u(t) \geq \int_0^t e^{(t-s)\triangle} s^p (e^{s\triangle} u_0)^{p^2} ds \geq \int_0^t s^p (e^{t\triangle} u_0)^{p^2} ds = \frac{t^{p+1}}{p+1} (e^{t\triangle} u_0)^{p^2}.$$

This is the case $k = 2$ of the following inequality which for any $k \in \mathbb{N}$ with $k \geq 2$ we will obtain by induction:

$$u(t) \geq \frac{t^{1+p+\ldots+p^{k-1}} (e^{t\triangle} u_0)^{p^k}}{(1+p)^{p^{k-2}} (1+p+p^2)^{p^{k-3}} \ldots (1+p+\ldots+p^{k-1})} = \frac{t^{\frac{p^k-1}{p-1}} (e^{t\triangle} u_0)^{p^k}}{\prod_{\ell=2}^k \left(\frac{p^\ell - 1}{p-1} \right)^{p^{k-\ell}}}. \tag{9.8}$$

Indeed, assuming (9.8) for k and repeating (9.7) we have

$$u(t) \geq \int_0^t e^{(t-s)\triangle} u^p(s) ds \geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k \left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}} e^{(t-s)\triangle}(e^{s\triangle}u_0)^{p^{k+1}} ds$$

$$\geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k \left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}} ds (e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^k-1}{p-1}p+1}}{\prod_{\ell=2}^k \left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}} \left(\frac{p^k-1}{p-1}p+1\right)} (e^{t\triangle}u_0)^{p^{k+1}}$$

$$= \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^k \left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}} \frac{p^{k+1}-1}{p-1}} (e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^{k+1} \left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}} (e^{t\triangle}u_0)^{p^{k+1}}.$$

So (9.8) holds also for $k+1$ and hence for any $k \in \mathbb{N}$ with $k \geq 2$. Then

$$t^{\frac{p^k-1}{(p-1)p^k}} e^{t\triangle}u_0 \leq (u(t))^{\frac{1}{p^k}} \prod_{\ell=2}^k \left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}} \Rightarrow t^{\frac{1}{p-1}} e^{t\triangle}u_0 \leq \prod_{\ell=2}^\infty \left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}}$$

$$= e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\frac{p^\ell-1}{p-1}\right)} = e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\sum_{j=1}^{\ell-1}p^j\right)} \leq e^{\sum_{\ell=2}^\infty p^{-\ell}\log(\ell p^\ell)} < +\infty.$$

This proves (9.5).

Proof of the case $p = 1 + \frac{2}{n}$ We return to the proof of Theorem 9.2 when $p = 1 + \frac{2}{n}$. If instead of looking at solutions in $C_0(\mathbb{R}^n)$ we look at solutions in $X := C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then our $u \in C^0([0, T_{u_0}), C_0(\mathbb{R}^n))$ is also $u \in C^0([0, T_{u_0}), X)$. Indeed, if the lifespan in X was shorter, then for some $t_0 < T_{u_0}$ we would have

$$\lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^n)} = \infty \text{ while } \sup_{0 \leq t \leq t_0} \|u(t)\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

But this is impossible because from (9.4) for $t < t_0$ we get

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)} + \int_0^t \|u(s)\|_{L^\infty(\mathbb{R}^n)}^{p-1} \|u(s)\|_{L^1(\mathbb{R}^n)} ds$$

implies by the Gronwall inequality

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)} e^{t_0 (\sup_{0 \leq t \leq t_0} \|u(t)\|_{L^\infty(\mathbb{R}^n)})^{p-1}} < \infty$$

and so

$$+\infty = \lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)} e^{t_0 (\sup_{0 \leq t \leq t_0} \|u(t)\|_{L^\infty(\mathbb{R}^n)})^{p-1}} < +\infty,$$

which is absurd.

Hence we conclude that $t_0 = T_{u_0}$ and we have $u \in C^0([0, T_{u_0}), L^1(\mathbb{R}^n))$, and so $u(t) \in L^1(\mathbb{R}^n)$ for all $t \in [0, T_{u_0})$. Since any such t can be taken as an initial value at time t for our solution, it follows that

$$\tau^{\frac{n}{2}}(e^{\tau\triangle}u(t))(x) \leq C \text{ for a fixed } C > 0 \text{ , any } x \in \mathbb{R}^n \text{ and } 0 < \tau < T_{u_0} - t$$

and for all $t \in [0, T_{u_0})$. In particular if $T_{u_0} = \infty$

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \leq (4\pi)^{\frac{n}{2}} C \text{ for all } t \geq 0. \tag{9.9}$$

Initially we assume that $u_0 \geq kK_\alpha$, for $K_\alpha(x) := (4\pi\alpha)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\alpha}}$. Notice that $K_\alpha = e^{\alpha\triangle}\delta_0$. Then we have (a bit formally, but can be checked)

$$u(t) \geq e^{t\triangle}u_0 \geq ke^{t\triangle}K_\alpha = ke^{t\triangle}e^{\alpha\triangle}\delta_0 = ke^{(\alpha+t)\triangle}\delta_0 = kK_{\alpha+t}.$$

Now we have

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \geq \| \int_0^t e^{(t-s)\triangle}u^p(s)ds\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} dx \int_0^t e^{(t-s)\triangle}u^p(s)(x)ds$$
$$= \int_0^t ds \int_{\mathbb{R}^n} dx e^{(t-s)\triangle}u^p(s)(x) = \int_0^t \|e^{(t-s)\triangle}u^p(s)\|_{L^1(\mathbb{R}^n)}ds \text{ (by commuting the order of integration)}$$
$$\geq \int_0^t \|e^{(t-s)\triangle}(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^n)}ds$$
$$= \int_0^t ds \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy K_{t-s}(x-y)(e^{s\triangle}u_0)^p(y) = \int_0^t ds \int_{\mathbb{R}^n} dy (e^{s\triangle}u_0)^p(y) \underbrace{\int_{\mathbb{R}^n} dx K_{t-s}(x-y)}_{1}$$
$$= \int_0^t \|(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^n)}ds \geq k^p \int_0^t \|(e^{s\triangle}K_\alpha)^p ds\|_{L^1(\mathbb{R}^n)} = k^p \int_0^t \|K^p_{\alpha+s}\|_{L^1(\mathbb{R}^n)}ds.$$

Now notice that since $p = 1 + 2/n$

$$K^p_\beta(x) = (4\pi\beta)^{-\frac{n}{2}p}e^{-\frac{p|x|^2}{4\beta}} = (4\pi\beta)^{-\frac{n}{2}(p-1)}p^{-\frac{n}{2}}(4\pi\beta/p)^{-\frac{n}{2}}e^{-\frac{p|x|^2}{4\beta}} = (4\pi\beta)^{-\frac{n}{2}(p-1)}p^{-\frac{n}{2}}K^p_{\frac{\beta}{p}}(x)$$
$$= (4\pi\beta)^{-1}p^{-\frac{n}{2}}K^p_{\frac{\beta}{p}}(x).$$

This implies that if by absurd we suppose $T_{u_0} = +\infty$ then we have

$$\|u(t)\|_{L^1(\mathbb{R}^n)} \geq p^{-\frac{n}{2}}k^p \int_0^t (4\pi(\alpha+s))^{-1}\|K_{\frac{\alpha+s}{p}}\|_{L^1(\mathbb{R}^n)}ds$$
$$= p^{-\frac{n}{2}}k^p(4\pi)^{-1} \int_0^t (\alpha+s)^{-1}ds \to +\infty \text{ as } t \nearrow \infty.$$

This contradicts (9.9).

Suppose now we don't have $u_0 \geq kK_\alpha$. Let us set $v(t) = u(t+\varepsilon)$ for some $\varepsilon > 0$. The $v(t)$ is a solution of (9.4) with initial value $u(\varepsilon)$. We have $u(\varepsilon) \geq e^{\varepsilon\triangle} u_0$

$$\begin{aligned} v(0) = u(\varepsilon) \geq e^{\varepsilon\triangle} u_0 &= (4\pi\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\varepsilon}} f(y)dy = (4\pi\varepsilon)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^n} e^{\frac{|x+y|^2}{4\varepsilon}} e^{-\frac{|y|^2}{2\varepsilon}} f(y)dy \\ &\geq (4\pi\varepsilon)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2\varepsilon}} f(y)dy = kK_{\frac{\varepsilon}{2}} \end{aligned}$$

where we used the parallelogram formula

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

But then $v(t)$ blows up in finite time, and so $u(t)$ does too. This completes the proof of Theorem 9.2 also in the case $p = 1 + \frac{2}{n}$.

□

So far we have proved the blow up when $1 < p \leq 1 + \frac{2}{n}$ for positive initial data with $u_0 \in C_0^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. But in fact the result holds for $u_0 \in C_0^0(\mathbb{R}^n)$ because of the maximum principle.

Lemma 9.4. *Suppose that* $0 \leq v_0 \leq u_0$ *are in* $C_0^0(\mathbb{R}^n)$ *and let* $u(t), v(t) \in C^0([0,T], C_0^0(\mathbb{R}^n))$ *be corresponding solutions of* (9.4). *Then* $u(t) \geq v(t)$.

This follows by Lemma 8.6 and means that if $u_0 \in C_0^0(\mathbb{R}^n)$ but $u_0 \notin L^1(\mathbb{R}^n)$, the conclusions of Theorem (9.2) continue to hold, because we can find a $0 \leq v_0 \leq u_0$ with $v_0 \in C_0^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and v_0 non zero whose corresponding $v(t)$ blows up. Then by the maximum principle also $u(t)$ blows up.

The coefficient $p = 1 + \frac{2}{n}$ is critical. In fact we have the following global existence result for small initial data.

Theorem 9.5. *Let* $p > 1 + \frac{2}{n}$ *and* $u_0 \in X := C_0^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. *There is an* $\epsilon_0 > 0$ *s.t. for* $\|u_0\|_X < \epsilon_0$ *then equation* (9.4) *admits a global solution in* $C^0([0,\infty), C_0^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$.

In the proof of Theorem 9.5 we will use the *Japanese bracket* $\langle t \rangle := \sqrt{1+t^2}$.
Proof of Theorem 9.5. (9.2) can be solved locally because $u \to |u|^{p-1}u$ is locally Lipschitz in $C_0^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Now we prove that it is globally defined if $\epsilon_0 > 0$ is sufficiently small. Suppose that the maximum interval of existence is $[0,T)$ and let $\tau \in (0,T)$. Then set

$$\begin{aligned} \|u\|_\infty^{(\tau)} &= \|\langle t \rangle^{\frac{n}{2}} u\|_{L^\infty([0,\tau], L^\infty(\mathbb{R}^n))} \\ \|u\|_1^{(\tau)} &= \|u\|_{L^\infty([0,\tau], L^1(\mathbb{R}^n))}. \end{aligned}$$

We will prove that there is a fixed constant C_0 and a function $F(x_1,x_2) = \sum c_{a,b}|x_1|^a|x_2|^b$ with $a+b > 1$ (the sums are finite and with fixed constants $c_{a,b} > 0$) s.t. we have

$$\begin{aligned} \|u\|_\infty^{(\tau)} &\leq C_0\epsilon_0 + F(\|u\|_\infty^{(\tau)}, \|u\|_1^{(\tau)}) \\ \|u\|_1^{(\tau)} &\leq C_0\epsilon_0 + F(\|u\|_\infty^{(\tau)}, \|u\|_1^{(\tau)}). \end{aligned} \tag{9.10}$$

Then, assume that we have $\|u\|_\infty^{(\tau)} \le 2C_0\epsilon_0$ and $\|u\|_1^{(\tau)} \le 2C_0\epsilon_0$, for ϵ_0 sufficiently small we can assume $|F| < \frac{C_0}{2}\epsilon_0$. So we can conclude that $\|u\|_\infty^{(\tau)} \le 2C_0\epsilon_0$ and $\|u\|_1^{(\tau)} \le 2C_0\epsilon_0$ imply

$$\|u\|_\infty^{(\tau)} \le \frac{3}{2}C_0\epsilon_0$$
$$\|u\|_1^{(\tau)} \le \frac{3}{2}C_0\epsilon_0.$$

Hence we conclude that for any $t < T$ we have $\langle t\rangle^{\frac{n}{2}}\|u(t)\|_{L^\infty} \le \frac{3}{2}\epsilon_0$ and $\|u(t)\|_{L^1} \le \frac{3}{2}\epsilon_0$. But if $T < \infty$ we have

$$\infty = \lim_{t\nearrow T}\|u(t)\|_{L^1\cap L^\infty} \le \frac{3}{2}\epsilon_0,$$

which is absurd.

So now we turn to the proof of (9.10). We can always assume by taking $\epsilon_0 > 0$ small that $T \gg 1$, so we can pick τ large. For $t \le 10$ we have for $j = 1, \infty$

$$\|u\|_j^{(\tau)} \le \langle 10\rangle^{\frac{n}{2}}\|u_0\|_{L^j} + \langle 10\rangle^{\frac{n}{2}}\int_0^t \|u(s)\|_{L^\infty}^{p-1}\|u(s)\|_{L^1}ds \le \langle 10\rangle^{\frac{n}{2}}\|u_0\|_{L^j} + 10\langle 10\rangle^{\frac{n}{2}}(\|u\|_\infty^{(\tau)})^{p-1}\|u\|_j^{(\tau)}.$$

For $t > 10$ we have

$$u(t) = e^{t\triangle}u_0 + \int_0^t e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds = e^{t\triangle}u_0 + \int_0^{\frac{t}{2}} e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds$$
$$+ \int_{\frac{t}{2}}^{t-1} e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds + \int_{t-1}^t e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds = I + II + III + IV.$$

Now for each $t \in [10, \tau]$ we bound the L^∞ norm of each term in the right hand side. We have

$$\|e^{t\triangle}u_0\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{n}{2}}\epsilon_0.$$

We have

$$\|II\|_{L^\infty} \le \int_0^{\frac{t}{2}} \|e^{(t-s)\triangle}\|_{L^1\to L^\infty}\|u(s)\|_{L^\infty}^{p-1}\|u(s)\|_{L^1}ds \lesssim \int_0^{\frac{t}{2}}\langle t-s\rangle^{-\frac{n}{2}}\langle s\rangle^{-\frac{n}{2}(p-1)}ds(\|u\|_\infty^{(\tau)})^{p-1}\|u\|_1^{(\tau)}$$
$$\le C'\langle t\rangle^{-\frac{n}{2}}\int_0^{\frac{t}{2}}\langle s\rangle^{-\frac{n}{2}(p-1)}ds(\|u\|_\infty^{(\tau)})^{p-1}\|u\|_1^{(\tau)} \le C\langle t\rangle^{-\frac{n}{2}}(\|u\|_\infty^{(\tau)})^{p-1}\|u\|_1^{(\tau)}$$

where we used $\frac{n}{2}(p-1) > 1$ (the latter equivalent to $p > 1 + \frac{2}{n}$). We have

$$\|IV\|_{L^\infty} \le \int_{t-1}^t \|e^{(t-s)\triangle}\|_{L^\infty\to L^\infty}\|u(s)\|_{L^\infty}^p ds \lesssim \int_{t-1}^t \langle s\rangle^{-\frac{n}{2}p}ds(\|u\|_\infty^{(\tau)})^p \le C\langle t\rangle^{-\frac{n}{2}}(\|u\|_\infty^{(\tau)})^p.$$

Finally to bound III there are 2 cases, either $p \geq 2$ which is easier, or $p < 2$. If $p \geq 2$ we bound

$$\|III\|_{L^\infty} \leq \int_{\frac{t}{2}}^{t-1} \|e^{(t-s)\triangle}\|_{L^1\to L^\infty} \|u(s)\|_{L^\infty}^{p-1} \|u(s)\|_{L^1} ds \lesssim \int_{\frac{t}{2}}^{t-1} \langle t-s\rangle^{-\frac{n}{2}} \langle s\rangle^{-\frac{n}{2}(p-1)} ds (\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)}$$

$$\leq C' \langle t\rangle^{-\frac{n}{2}(p-1)} \int_{\frac{t}{2}}^{t-1} \langle s-t\rangle^{-\frac{n}{2}} ds (\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)} \leq C\langle t\rangle^{-\frac{n}{2}(p-1)} (\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)} \begin{cases} \langle t\rangle^{1-\frac{n}{2}} \text{ if } n=1 , \\ \log(2+\langle t\rangle) \text{ if } n=2 \\ 1 \text{ if } n>2 \end{cases}$$

$$\leq C\langle t\rangle^{-\frac{n}{2}} (\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)}.$$

Notice that we used the fact that for $n = 1$ we have $p > 3$ and for $n = 2$ we have $p > 2$. Notice also that for $p < 2$ this argument does not give us the desired result.
If $p < 2$ (necessarily $n \geq 3$) we consider $\frac{1}{q} = p - 1$ and the corresponding $\frac{1}{q'} = 2 - p$, and

$$\|III\|_{L^\infty} \leq \int_{\frac{t}{2}}^{t-1} \|e^{(t-s)\triangle}\|_{L^q\to L^\infty} \|u(s)\|_{L^\infty}^{p-1} \|u(s)\|_{L^q} ds$$

$$\lesssim \int_{\frac{t}{2}}^{t-1} \langle t-s\rangle^{-\frac{n}{2q}} \|u(s)\|_{L^\infty}^{p-1+\frac{1}{q'}} \|u(s)\|_{L^1}^{\frac{1}{q}} ds = \int_{\frac{t}{2}}^{t-1} \langle t-s\rangle^{-\frac{n}{2q}} \|u(s)\|_{L^\infty} \|u(s)\|_{L^1}^{\frac{1}{q}} ds$$

$$\leq \int_{\frac{t}{2}}^{t-1} \langle t-s\rangle^{-\frac{n}{2}(p-1)} \langle s\rangle^{-\frac{n}{2}} ds \|u\|_\infty^{(\tau)} (\|u\|_1^{(\tau)})^{p-1} \leq C'\langle t\rangle^{-\frac{n}{2}} \int_{\frac{t}{2}}^{t-1} \langle t-s\rangle^{-\frac{n}{2}(p-1)} ds \|u\|_\infty^{(\tau)} (\|u\|_1^{(\tau)})^{p-1}$$

$$\leq C\langle t\rangle^{-\frac{n}{2}} \|u\|_\infty^{(\tau)} (\|u\|_1^{(\tau)})^{p-1}.$$

A comment on this last computation. Since the previous computation could not possibly yield the desired result, we have succeed by sacrificing some of the factor $\langle t-s\rangle^{-\frac{n}{2}}$, replacing it with $\langle t-s\rangle^{-\frac{n}{2}(p-1)}$, which however is good enough, but gaining in this way the fact that instead of $\|u(s)\|_{L^\infty}^{p-1}$ we get the better term $\|u(s)\|_{L^\infty}^{p-1+\frac{1}{q'}} = \|u(s)\|_{L^\infty}$, which is exactly what we need to get the factor $\langle s\rangle^{-\frac{n}{2}} \sim \langle t\rangle^{-\frac{n}{2}}$.

We also have

$$\|u(t)\|_{L^1} \leq \|e^{t\triangle}u_0\|_{L^1} + \int_0^t \|e^{(t-s)\triangle}\|_{L^1\to L^1} \|u(s)\|_{L^\infty}^{p-1} \|u(s)\|_{L^1} ds$$

$$\leq \epsilon_0 + \int_0^t \langle s\rangle^{-\frac{n}{2}(p-1)} ds (\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)} \leq \epsilon_0 + C(\|u\|_\infty^{(\tau)})^{p-1} \|u\|_1^{(\tau)}$$

from $\frac{n}{2}(p-1) > 1$.

□

9.2 Global well posedness

We consider now instead the Cauchy problem for the heat equation

$$\begin{cases} u_t = \triangle u - |u|^{p-1}u \text{ with } (t,x) \in (0,T)\times\mathbb{R}^n \text{ and } p>1, \\ u(0,x) = u_0(x) \text{ where } u_0 \in C_0(\mathbb{R}^n,\mathbb{R}). \end{cases} \tag{9.11}$$

We can apply the abstract theory on semilinear equations and conclude that for any $u_0 \in X$ there is a maximal $T_{u_0} \in (0, +\infty]$ and a unique $u(t) \in C^0([0, T_{u_0}), C_0(\mathbb{R}^n, \mathbb{R}))$ satisfying

$$u(t) = e^{t\triangle}u_0 - \int_0^t e^{(t-s)\triangle}|u(s)|^{p-1}u(s)ds. \tag{9.12}$$

For $\epsilon > 0$ let us consider $g_\epsilon(|u|^2) = (\epsilon + |u|^2)^{\frac{p-1}{2}}$ and $F_\epsilon(u) = g_\epsilon(|u|^2)u$ and the equation

$$u(t) = e^{t\triangle}u_0 - \int_0^t e^{(t-s)\triangle}F_\epsilon(u(s))ds. \tag{9.13}$$

Lemma 9.6. *Let $u(t) \in C^0([0,T), C_0(\mathbb{R}^n))$ be a solution of* (9.13) *and suppose that $u_0 \in C_0^m(\mathbb{R}^n)$. Then $u(t) \in C([0,T), C_0^m(\mathbb{R}^n))$.*

Proof. First of all, if $u_0 \in C_0^m(\mathbb{R}^n)$ then $e^{t\triangle}u_0 \in C([0,\infty), C_0^m(\mathbb{R}^n))$ and furthermore $\|e^{t\triangle}u_0\|_{W^{m,\infty}(\mathbb{R}^n)} \le \|u_0\|_{W^{m,\infty}(\mathbb{R}^n)}$. When we solve (9.13) in $C_0(\mathbb{R}^n)$, we consider a fixed point problem in

$$E = \{u \in C^0([0,T_M], C_0(\mathbb{R}^n)) : \|u(t)\|_\infty \le 2\|u_0\|_\infty \text{ for all } t \in [0,T_M]\}$$

for $M = \|u_0\|_\infty$ and $T_M = \frac{1}{2L(2M)}$. If we pick $u \in C^0([0,T_M], C_0^m(\mathbb{R}^n))$ and if we set

$$\Phi_u(t) = e^{t\triangle}u_0 - \int_0^t e^{(t-s)\triangle}F_\epsilon(u(s))ds,$$

then $\Phi_u \in C^0([0,T_M], C_0^m(\mathbb{R}^n))$ with, by the chain rule,

$$\partial_x^\alpha \Phi_u(t) = e^{t\triangle}\partial_x^\alpha u_0 - \sum_{k=1}^{|\alpha|} \sum_{|\beta_1|+...+|\beta_1|=|\alpha|} c_{\alpha\beta} \int_0^t e^{(t-s)\triangle}F_\epsilon^{(k)}(u(s))\partial_x^{\beta_1}u(s)...\partial_x^{\beta_k}u(s)ds$$

for appropriate constants $c_{\alpha\beta} = c_{\alpha\beta_1,...,\beta_k}$. Notice that the only element where the summation on the r.h.s. depends on a derivative of order $|\alpha|$ is

$$-\int_0^t e^{(t-s)\triangle}F'_\epsilon(u(s))\partial_x^\alpha u(s)ds.$$

For any A let $L'(A)$ be such that for any $u, v \in C_0^m(\mathbb{R}^n)$ with $\|u\|_{W^{m,\infty}(\mathbb{R}^n)} \le A$ and $\|v\|_{W^{m,\infty}(\mathbb{R}^n)} \le A$ we have

$$\sum_{|\alpha|\le m} \| \sum_{k=1}^{|\alpha|} \sum_{|\beta_1|+...+|\beta_k|=|\alpha|} c_{\alpha\beta}[F_\epsilon^{(k)}(u)\partial_x^{\beta_1}u...\partial_x^{\beta_k}u - F_\epsilon^{(k)}(v)\partial_x^{\beta_1}v...\partial_x^{\beta_k}v]\|_\infty \le L'(A)\|u-v\|_{W^{m,\infty}(\mathbb{R}^n)}.$$

Set $M' = 2\|u_0\|_{W^{m,\infty}(\mathbb{R}^n)}$. Then for $T' = \frac{1}{2L'(2M')}$ consider

$$E_m = \{u \in C^0([0,T'], C_0^m(\mathbb{R}^n)) : \|u(t)\|_{W^{m,\infty}(\mathbb{R}^n)} \le M' \text{ for all } t \in [0,T_M]\}.$$

It is then easy to see that $u \to \Phi_u$ preserves E_m and is a contraction therein. So there is a fixed point and hence a solution $u \in C^0([0,T'], C_0^m(\mathbb{R}^n))$ of (9.13), which is obviously the solution in $C^0([0,T'], C_0^0(\mathbb{R}^n))$. Let now consider the maximal solution $u(t) \in C([0,T), C_0(\mathbb{R}^n))$ and the maximal solution $u(t) \in C([0,T_m), C_0^m(\mathbb{R}^n))$. Evidently $T_m \le T$, and we claim that $T_m = T$.
Let us consider case $m = 1$. We have

$$\partial_x^\alpha u(t) = e^{t\triangle}\partial_x^\alpha u_0 - \int_0^t e^{(t-s)\triangle} F'_\epsilon(u(s))\partial_x^\alpha u(s)ds.$$

from which we see that by Gronwall

$$\|\partial_x^\alpha u(t)\|_\infty \le \|\partial_x^\alpha u_0\|_\infty + \int_0^t \|F'_\epsilon(u(s))\|_\infty \|\partial_x^\alpha u(s)\|_\infty ds \Rightarrow \|\partial_x^\alpha u(t)\|_\infty \le \|\partial_x^\alpha u_0\|_\infty e^{\int_0^t \|F'_\epsilon(u(s))\|_\infty ds}$$

so that we cannot have $\|\partial_x^\alpha u(t)\|_\infty \overset{t\to T_1}{\to} \infty$ if $T_1 < T$. Suppose now we have shown $T_{m-1} = T$. By a similar method we show that $T_m = T$. Indeed we have for any $|\alpha| = m$

$$\partial_x^\alpha u(t) = c_\alpha(t) - \int_0^t e^{(t-s)\triangle} F'_\epsilon(u(s))\partial_x^\alpha u(s)ds \text{ where}$$
$$c_\alpha(t) = -\sum_{k=2}^{|\alpha|} \sum_{|\beta_1|+...+|\beta_k|=|\alpha|} c_{\alpha\beta} \int_0^t e^{(t-s)\triangle} F_\epsilon^{(k)}(u(s))\partial_x^{\beta_1}u(s)...\partial_x^{\beta_k}u(s)ds.$$

Since $c_\alpha(t)$ depends on derivatives of order $\le m-1$ we have $c_\alpha(t) \in C^0([0,T), C_0(\mathbb{R}^n))$. Then we conclude $T_m = T$ by the same argument as for the case $m = 1$. $\square$

The solutions of (9.13) satisfy the following.

Lemma 9.7. *Let $u \in C([0,T), C_0(\mathbb{R}^n, \mathbb{R}))$ be a solution of* (9.13) *and let $u_0 \ge 0$. Then $u(t,x) \ge 0$ for all $(t,x) \in [0,T) \times \mathbb{R}^n$.*

Proof. First of all, by well posedness it is enough to consider just $u_0 \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$, as we will see below. If $u_0 \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ then by Lemma 9.6 we have $u(t) \in C([0,T), C_0^m(\mathbb{R}^n))$ for all m. Then $u(t)$ solves not only the integral equation (9.13), but by Corollary 7.4 (claim (i)) solves also the differential equation:

$$u_t = \triangle u - g_\epsilon(|u|^2)u.$$

Then $u(t) \in C^1([0,T), C_0^m(\mathbb{R}^n))$ for all m.
Let us assume that $u_0 \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ with $u_0 \ge 0$ exists s.t. there is a $t_0 > 0$ such that $0 > -\mu := \inf_{x\in\mathbb{R}^n} u(t_0, x)$. Let

$$t_1 = \inf\{t \in (0, t_0] : \inf_{x\in\mathbb{R}^n} u(t,x) = -\mu\}.$$

Then, $t_1 \leq t_0$ and since $u \in C([0,t_0], C_0(\mathbb{R}^n))$ we have $t_1 > 0$. Let x_1 be a point of minimum of $u(t_1,x)$. Then $\nabla_x u(t_1,x_1) = 0$, the Hessian $H(t_1,x_1)$ of u is positive definite and so $\triangle u(t_1,x_1) = \mathrm{trace} H(t_1,x_1) \geq 0$. Then we have

$$0 \geq \partial_t u(t_1,x_1) \geq -g_\epsilon(|u(t_1,x_1)|^2)u(t_1,x_1) = \langle \epsilon + \mu \rangle^{\frac{p-1}{2}} \mu > 0.$$

This is absurd and so Lemma 9.7 holds if $u_0 \in C_c^\infty(\mathbb{R}^n)$. In the general case let $C_c^\infty(\mathbb{R}^n,\mathbb{R}) \ni u_\nu(0,x) \stackrel{\nu\to\infty}{\to} u_0(0,x)$ in $C_0(\mathbb{R}^n,\mathbb{R})$ and with $u_\nu(0,x) \geq 0$. Then by well posedness we have $u_\nu(t,x) \stackrel{\nu\to\infty}{\to} u(t,x)$ in $C_0(\mathbb{R}^n,\mathbb{R})$ and so $u(t,x) \geq 0$. □

Lemma 9.8. *Let $u, v \in C([0,T), C_0(\mathbb{R}^n,\mathbb{R}))$ be solutions of (9.13) and let $u_0 \geq v_0$ for their initial data. Then $u(t,x) \geq v(t,x)$ for all $(t,x) \in [0,T) \times \mathbb{R}^n$.*

Proof. Again it is enough to consider just $u_0, v_0 \in C_c^\infty(\mathbb{R}^n,\mathbb{R})$. Let us assume that $u_0, v_0 \in C_c^\infty(\mathbb{R}^n,\mathbb{R})$ with $u_0 \geq v_0$ exist s.t. there is a $t_0 > 0$ such that $0 > -\mu := \inf_{x\in\mathbb{R}^n} w(t_0,x)$ where $w(t,x) := u(t,x) - v(t,x)$. Let

$$t_1 = \inf\{t \in (0,t_0] : \inf_{x\in\mathbb{R}^n} w(t,x) = -\mu\}.$$

Then, $0 < t_1 \leq t_0$ like before. Let x_1 be a point of minimum of $w(t_1,x)$. Then $\triangle w(t_1,x_1) \geq 0$. Notice that we have $w(t_1,x_1) = -\mu$, so in particular $u(t_1,x_1) < v(t_1,x_1)$ and so $g_\epsilon(|v(t_1,x_1)|^2)v(t_1,x_1) > g_\epsilon(|u(t_1,x_1)|^2)u(t_1,x_1)$ by the fact that $t \to g_\epsilon(t^2)t = (\epsilon + t^2)^{\frac{p-1}{2}} t$ is strictly increasing. So we have

$$0 \geq \partial_t w(t_1,x_1) \geq -g_\epsilon(|u(t_1,x_1)|^2)u(t_1,x_1) + g_\epsilon(|v(t_1,x_1)|^2)v(t_1,x_1) > 0.$$

This is absurd and so Lemma 9.8 holds if $u_0, v_0 \in C_c^\infty(\mathbb{R}^n)$. The general case follows by density. □

Corollary 9.9. *Let $u \in C([0,T_{u_0}), C_0(\mathbb{R}^n,\mathbb{R}))$ be the maximal solution of (9.13). Then $T_{u_0} = \infty$.*

Proof. If $T_{u_0} < \infty$ then $\lim_{t\nearrow T_{u_0}} \|u(t)\|_{L^\infty} = +\infty$. In the case $u_0 \geq 0$ we have we have for all $t \in [0,T_{u_0})$

$$0 \leq u(t) = e^{t\triangle}u_0 - \int_0^t e^{(t-s)\triangle}(\epsilon + |u^{(\epsilon)}(s)|^2)^{\frac{p-1}{2}} u(s)ds \leq e^{t\triangle}u_0 \leq \|u_0\|_{L^\infty} < \infty.$$

then $T_{u_0} = \infty$. Suppose now that u_0 does not have constant sign. Then we have $-|u_0| \leq u_0 \leq |u_0|$ and let $v(t) \in C([0,\infty), C_0(\mathbb{R}^n,\mathbb{R}))$ be the solution with $v(0) = |u_0|$. Then $-v(t) \leq u(t) \leq v(t)$ and this implies $T_{u_0} = \infty$.

□

Lemma 9.10. *Let $u \in C([0,T_{u_0}), C_0(\mathbb{R}^n,\mathbb{R}))$ the maximal solution of (9.12) and let $u^{(\epsilon)} \in C([0,\infty), C_0(\mathbb{R}^n,\mathbb{R}))$ be the solutions of (9.13). Then for any $0 < T < T_{u_0}$ we have $u^{(\epsilon)} \stackrel{\epsilon\to 0}{\to} u$ in $C([0,T], C_0(\mathbb{R}^n,\mathbb{R}))$.*

Proof. We have

$$u^{(\epsilon)}(t) - u(t) = -\int_0^t e^{(t-s)\triangle}\left[((\epsilon + |u^{(\epsilon)}(s)|^2)^{\frac{p-1}{2}} - |u^{(\epsilon)}(s)|^{p-1}\right] u^{(\epsilon)}(s)$$
$$-\int_0^t e^{(t-s)\triangle}\left[|u^{(\epsilon)}(s)|^{p-1}u^{(\epsilon)}(s) - |u(s)|^{p-1}u(s)\right].$$

Now we have $\|u^{(\epsilon)}(s)\|_\infty \le \|u_0\|_\infty$ by the discussion in Corollary 9.9. Using this fact we have also for ϵ small

$$\|(\epsilon + |u^{(\epsilon)}(s)|^2)^{\frac{p-1}{2}} - |u^{(\epsilon)}(s)|^{p-1}\|_\infty \le \sup_{|t|\le\|u_0\|_\infty} |(\epsilon + |t|^2)^{\frac{p-1}{2}} - |t|^{p-1}| \le C_{\|u_0\|_\infty}\epsilon^{\min(\frac{p-1}{2},1)}.$$

Indeed if we set $\varphi(s) = (\epsilon + s)^\alpha - s^\alpha$ for $s \in [0, M]$ and $\alpha > 0$ we have $\varphi'(s) = \alpha(\epsilon + s)^-1 - \alpha s^{\alpha-1}$ and this has constant sign, so that $\varphi(s) \le \max(\varphi(0), \varphi(M))$. Now $\varphi(0) = \epsilon^\alpha$ and

$$\varphi(M) = M^\alpha[(1 + \epsilon/M + s)^\alpha - 1] = M^\alpha[\epsilon/M + O(\epsilon^2)].$$

Then for $t \in [0, T]$

$$\|u^{(\epsilon)}(t) - u(t)\|_\infty \le TC_{\|u_0\|_\infty}\epsilon^{\min(\frac{p-1}{2},1)}\|u_0\|_\infty + L(M)\int_0^t \|u^{(\epsilon)}(s) - u(s)\|_\infty,$$

with $M = \max\{\|u_0\|_\infty, \sup_{s\in[0,T]} \|u(s)\|_\infty\}$. Then by Gronwall for $t \in [0, T]$

$$\|u^{(\epsilon)}(t) - u(t)\|_\infty \le TC_{\|u_0\|_\infty}\epsilon^{\min(\frac{p-1}{2},1)}\|u_0\|_\infty e^{TL(M)} \overset{\epsilon\to 0}{\to} 0 \text{ uniformly in } [0, T].$$

□

Lemma 9.11. *Let $u, v \in C([0, T_*), C_0(\mathbb{R}^n, \mathbb{R}))$ be solutions of* (9.12) *with $u_0 \ge v_0$. Then $u(t, x) \ge v(t, x)$ for all $(t, x) \in [0, T_*) \times \mathbb{R}^n$.*

Proof. It is enough to show this for all $(t, x) \in [0, T] \times \mathbb{R}^n$ with $0 < T < T_*$. But we know $u^{(\epsilon)}(t, x) \ge v^{(\epsilon)}(t, x)$ for all $(t, x) \in [0, \infty) \times \mathbb{R}^n$. The desired result follows taking $\epsilon \searrow 0$.

Corollary 9.12. *Let $u \in C([0, T_{u_0}), C_0(\mathbb{R}^n, \mathbb{R}))$ be the maximal solution of* (9.12)*. Then $T_{u_0} = \infty$.*

Proof. It follows from Lemma 9.11 the same way Corollary 9.9 follows from Lemma 9.8.

□

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