Università degli Studi di Trieste

Dipartimento di Scienze Matematiche

QUADERNI MATEMATICI

II SERIE

GIOVANNI ALESSANDRINI

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Quaderno n. 276, agosto 1992

Edizione fuori commercio

SONO STATI ADEMPIUTI GLI OBBLIGHI DI LEGGE
(D.L.Lgt. 31 agosto 1945 n. 660)

Trieste, Università degli Studi
Dipartimento di Scienze Matematiche

A SIMPLE PROOF OF THE UNIQUE CONTINUATION PROPERTY FOR TWO DIMENSIONAL ELLIPTIC EQUATIONS IN DIVERGENCE FORM

Giovanni Alessandrini
Dipartimento di Scienze Matematiche
Università degli Studi di Trieste

1. Introduction

We consider weak solutions u of

$$div\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\nabla u\right) = 0 \ , \tag{1.1}$$

in a connected open set Ω of $\mathbb{R}^2$, with coefficients $a,b,c,d\in L^{\infty}(\Omega)$ satisfying the uniform ellipticity condition

$$\lambda(\xi^2+\eta^2)\leq a\xi^2+(b+c)\xi\eta+d\eta^2\leq\lambda^{-1}(\xi^2+\eta^2) \ , \text{ for every } (\xi,\eta)\in\mathbb{R}^2 \ , \tag{1.2}$$

where λ , $0<\lambda\leq 1$, is a given constant. We prove the following unique continuation theorem.

Theorem 1.1 *Let* $(x^0,y^0)\in\Omega$. *If* $u\in W^{1,2}_{loc}(\Omega)$ *is a solution of* (1.1) *and if, for every positive integer* N , *there exists a positive number* C_N *such that*

$$\int_{\Omega\cap B_R(x^0,y^0)} |\nabla u|^2 dxdy\leq C_N R^N \ , \text{ for every } R>0 \ , \tag{1.3}$$

then u *is constant in* Ω .

Our approach is based on proving that u is the first component of a quasiregular mapping $f=u+iv$ and showing that if ∇u vanishes of infinite order at (x^0,y^0) , in the average sense expressed by (1.3), then $f-f(x^0,y^0)$ vanishes of infinite order at the same point, thus yielding that f must be constant. The approach through quasiconformal mappings, and more generally, complex analytic methods, to the treatment of elliptic equations in two variables has been known for a long time and it has proven useful in many situations. More specifically, with regards to theorems about unique continuation, let us recall that Bers and Nirenberg [B-N] proved that, for nondivergence elliptic equations with bounded measurable coefficients, the gradient of any solution is a quasiregular mapping.

Curiosly enough, it appears that this method has not been applied to the equations under consideration here. Let us mention that, in case the coefficients a,b,c,d in (1.1) are Hölder continuous, and $b=c$, the unique continuation property has been proven by Schulz [S] quite recently.

On the other hand, the validity of the unique continuation property for equation (1.1) with discontinuous coefficients is particularly useful in the treatment of the inverse problem of determining the location of jumps in the coefficients when one knows Cauchy data for one specific solution.

More specifically, consider a subdomain D of Ω, suppose that ∂D is C^2-smooth and suppose that the coefficients in (1.1) are given by $a=d=1+\chi_D$, $b=c=0$. One would like to determine D by the knowledge of $u|_{\partial\Omega}$ and $\partial_\nu u|_{\partial\Omega}$ for one nontrivial solution u of (1.1), here ∂_ν denotes the normal derivative. For such a problem, local uniqueness theorems have been recently proven by Bellout, Friedman and Isakov [B-F-I] and by Powell [P] provided that ∂D is an analytic closed curve. But, in these papers, it is evident that, if the unique continuation property would hold, then the analiticity hypothesis could be removed. Theorem 1.1 above provides then the appropriate tool.

Ackowledgements. This problem was proposed to the author by Victor Isakov. The author wishes to thank him for the encouragement received.

2. Proof of Theorem 1.1

With no loss of generality we can assume that $(x^0,y^0)=0$, Ω is a disk $B_{R_0}(0)$ for some positive R_0, and also that $u\in W^{1,2}(\Omega)$, $u(0)=0$.

We start by constructing a function v associated to u in a way that generalizes the notion of conjugate harmonic function, we might call v the stream function associated to u, in analogy with the theory of gas dynamics, see for instance Bergman and Schiffer [B-S]. Equation (1.1) can be rephrased by saying that the 1-form $\omega=-(cu_x+du_y)dx+(au_x+bu_y)dy$ is closed in Ω and therefore we can find $v\in W^{1,2}(\Omega)$ such that $\omega=dv$. In other words, u and v satisfy the following elliptic system

(2.1a) $$v_x=-cu_x-du_y\ ,$$

(2.1b) $$v_y=au_x+bu_y\ ,$$

notice that this one is nothing but a special case of the elliptic systems studied by Bers and Nirenberg [B-N]. Thus we can follow their computations, and, by setting $z=x+iy$, $f=u+iv$, (2.1) takes the form

(2.2a) $$f_{\bar z}=\mu f_z+\nu\overline{(f_z)}\ ,$$

where

(2.2b) $$\mu=\frac{d-a-i(b+c)}{1+a+d+ad-bc}\ ,\quad \nu=\frac{1-ad+bc+i(b-c)}{1+a+d+ad-bc}\ ,$$

and the following estimate can be easily obtained

(2.3) $$|\mu|+|\nu|\le\frac{(1+\lambda)^2-2\lambda^3}{(1+\lambda)^2}<1\ .$$

Consequently, being $f\in W^{1,2}(\Omega,\mathbb{C})$, (2.2) tells us that f is a quasiregular mapping, or, as is the same, a quasiconformal function, see for instance, Lehto and Virtanen [L-V]. For the sake of rigor, the procedure we have just outlined should be preceded by a regularization of the coefficients and then by considering u as a classical solution, finally, by a passage to the limit, one could show that the whole construction above is justified also when no additional smoothness assumption is made. We omit the details.

Notice also that (2.1) implies that v is a solution of the following equation

(2.4) $$div\Big(\frac{1}{ad-bc}\begin{pmatrix} a & c\\ b & d\end{pmatrix}\nabla v\Big) = 0\ ,$$

which is too uniformly elliptic, with ellipticity constant λ .

Now we show that if (1.3) holds for all N then f has to be a constant. We make use of the local boundedness estimate for (1.1) and (2.4), see [G-T, Theorem 8.17], along with the Poincaré inequality. There exists A , only depending on λ , such that, for every $R\le R_0$ we have

(2.5) $$\max_{B_{R/2}(0)}|u-u_R|^2\le A\int_{B_R(0)}|\nabla u|^2\,dxdy\ ,$$

here

$$u_R=\frac{1}{|B_R|}\int_{B_R(0)}u\,dxdy\ ,$$

and the analogous estimate holds for v .

We may set $v(0)=0$. From (1.3), (2.1) and (2.5) we have that u_R and v_R vanish of infinite order as $R \to 0$. Applying once more (2.5) we obtain

(2.6)
$$\max_{B_{R/2}(0)} |f|^2 \leq 8AC_N R^N \ .$$

If f were not identically zero, then the well-known factorization theorem of quasiregular mappings, see [L-V], gives us that there exists ρ , $0<\rho<R_0$, a quasiconformal mapping $\chi: B_\rho(0) \to B_\rho(0)$, with $\chi(0)=0$, and a positive integer M such that

$$f(z)=\left[\chi(z)\right]^M \text{ for every } z\in B_\rho(0) \ .$$

Being χ^{-1} Hölder continuous with some exponent $\alpha \leq 1$, for some positive constant K we have

(2.7)
$$|f(z)| \geq K|z|^{M/\alpha} \text{ for every } z\in B_\rho(0) \ .$$

Now (2.7) gives a contradiction with (2.6) and hence f vanishes identically. □

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