Global stability for coupled physics inverse problems. A case study

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Introduction

Since the ’80s, a dominant theme in Inverse Problems has been:
To image the interior of an object from measurements taken in its exterior

- overdetermined boundary data,
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Interior data may provide much better stability than inverse boundary problems, or inverse scattering problems.
Available results require nondegeneracy conditions on the solutions of the involved direct problems:

- Nonvanishing of solution.
- Nonvanishing of gradients.
- Nonvanishing of Jacobians.
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Global stability

Question. Is it possible to obtain global stability from measurements arising from arbitrary (nontrivial) solutions of the direct problem?

The model problem A problem arising in microwave imaging coupled with ultrasound, Triki (2010).

$$\Delta u + qu = 0 \text{ in } \Omega$$

Find $q \geq$ constant > 0 given the local energy qu^2 and the boundary data $u|_{\partial\Omega}$.

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The full problem

Ammari, Capdeboscq, De Gournay, Rozanova-Pierrat, Triki (2011):

$$div(a\nabla u) + k^2 qu = 0 \text{ in } \Omega$$

Find $a, q \geq$ constant > 0 given the local energies qu^2 , $a|\nabla u|^2$ (with several u's and k's!).

Here:
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An example

In dimension $n = 1$, fix $0 < r < R$ and, for every $k = 1, 2, \ldots$, set

$$q_k(x) = \begin{cases} A_k & \text{if } |x| < r\,, \\ 1 & \text{if } r \le |x| \le R\,, \end{cases}$$

where

$$A_k = \left(\frac{\pi}{2} + 2k\pi\right)^2 r^{-2}\,.$$

A solution to $u_{xx} + q_k u = 0$ in $(-R, R)$ is

$$u_k(x) = \begin{cases} \frac{1}{\sqrt{A_k}} \cos(\sqrt{A_k}x) & \text{if } |x| < r\,, \\ -\sin(|x| - r) & \text{if } r \le |x| \le R\,. \end{cases}$$

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we have

$$\|q_{2k}u_{2k}^2 - q_k u_k^2\|_\infty \leq 2$$

whereas, for any p , $1 \leq p \leq \infty$

$$\|q_{2k} - q_k\|_p \to \infty \ .$$

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A priori assumptions

Given a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary (in quantitative form!), we consider one solution $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ to

$$\Delta u + qu = 0 \text{ in } \Omega$$

where $q \in L^\infty(\Omega)$ is assumed to satisfy

$$0 < K^{-1} \leq q \leq K \text{ a.e. in } \Omega$$

for a given $K \geq 1$.

A priori assumptions

Energy bound. $E > 0$ is given such that:

$$\int_\Omega |\nabla u|^2 + u^2 \leq E^2 \ .$$

Nontriviality of the data. $H > 0$ is given such that:

$$\int_\Omega q u^2 \geq H^2 > 0 \ .$$

A priori data: K, E, H and Ω (diam(Ω), constants of its Lipschitz character).

Notation: For every $d > 0$: $\Omega_d = \{x \in \Omega | dist(x, \partial\Omega) > d\}$.

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The main Theorem

Theorem. Let q_1, q_2 and the corresponding solutions u_1, u_2 satisfy the a priori assumptions and suppose that

$$\|q_1 u_1^2 - q_2 u_2^2\|_{L^\infty(\Omega)} \leq \varepsilon \, , \qquad (1)$$

for a given $\varepsilon > 0$, and also

$$\| |u_1| - |u_2| \|_{L^\infty(\partial\Omega)} \leq \sqrt{K\varepsilon} \, . \qquad (2)$$

Then, for every $d > 0$, there exists $\eta \in (0, 1)$ and $C > 0$, only depending on d and on the a priori data such that

$$\|q_1 - q_2\|_{L^2(\Omega_d)} \leq C \left(\varepsilon^{1/3} + \varepsilon \right)^{\eta} \, ,$$

Note: If we know $q_1 = q_2$ near $\partial\Omega$, then (1) $\Rightarrow$ (2).

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Main tools

Theorem A (Stability for $|u|$)
There exists $C > 0$, only depending on K, E and Ω, such that

$$\int_{\Omega} ||u_1| - |u_2||^3 \leq C\varepsilon \ .$$

Theorem B (Integrability of $|u|^{-\delta}$) For every $d > 0$, there exists $p > 1, C > 0$, only depending on K, E, H and Ω, such that

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Proof of the main theorem

$$(q_1 - q_2)u_1^2 = q_2(u_2^2 - u_1^2) + (q_1 u_1^2 - q_2 u_2^2) =$$
$$= q_2(|u_2| + |u_1|)(|u_2| - |u_1|) + (q_1 u_1^2 - q_2 u_2^2)$$

hence

$$\int_\Omega |q_1 - q_2| u_1^2 \leq K \| |u_1| + |u_2| \|_{L^{3/2}(\Omega)} \| |u_1| - |u_2| \|_{L^3(\Omega)} + |\Omega| \varepsilon$$

and, by Theorem A,

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and choosing $\delta = \frac{2}{p-1}$, by Theorem B

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Recalling $K^{-1} \leq q_i \leq K$ we arrive at

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Stability for $|u|$

Denote $N_i = \{u_i = 0\}$, $i = 1, 2$. Let Ω_j , $j = 1, 2, \dots$ be the connected components of $\Omega \setminus (N_1 \cup N_2)$.

For each j we may split

$$\partial\Omega_j = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \ ,$$

where

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By assumption, on Γ_0 we have $||u_1| - |u_2|| \leq \sqrt{K\varepsilon}$, while, on Γ_1, $q_2 u_2^2 \leq \varepsilon$ and on Γ_2, $q_1 u_1^2 \leq \varepsilon$. Hence, on $\partial\Omega_j$ we have

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$$\varphi^+ = \left[u_1 - u_2 - 2\sqrt{K\varepsilon}\right]^+ , \varphi^- = \left[u_2 - u_1 - 2\sqrt{K\varepsilon}\right]^+ .$$

Note that $\varphi^\pm \in W_0^{1,2}(\Omega_j) \cap C(\overline{\Omega_j})$ and use $\psi_i^\pm = \varphi^\pm u_i$ as test functions in the weak formulation of $\Delta u_i + q_i u_i = 0$. We arrive at

$$\int_{\Omega_j} (|u_1| + |u_2|)(|u_1| - |u_2|)^2 \leq C \int_{\Omega_j} (|u_1| + |u_2|)\varepsilon ,$$

Adding up w.r.t. j, and using the energy bound,

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Integrability of $|u|^{-\delta}$

Lipschitz propagation of smallness (A. and Rosset '98, A., Rondi, Rosset and Vessella 2009) If

$$\frac{\int_\Omega |\nabla u|^2 + u^2}{\int_\Omega u^2} \leq \mathcal{F}$$

then for any $B_\rho(x_0) \subset \Omega$ we have

$$\int_{B_\rho(x_0)} u^2 \geq C \int_\Omega |\nabla u|^2 + u^2$$

where $C > 0$ only depends on ρ, K, Ω and on $\mathcal{F}$.

Note: Under our a priori assumptions:

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Doubling inequality (Garofalo and Lin '86) There exists $R = R(K)$ such that if $r_0 \leq R$ and $B_{r_0}(x_0) \subset \Omega$ then

$$\int_{B_{2r}(x_0)} u^2 \leq C \int_{B_r(x_0)} u^2 \ , \forall r < \frac{r_0}{4}$$

where $C > 0$ only depends on r_0 and on the a priori data.

Integrability of $|u|^{-\delta}$

A_p **property** (Garofalo and Lin '86)
For any $d > 0$ there exist $p > 1, C > 0$, only depending on d and on the a priori data, such that for every $x_0 \in \Omega_d$ and every $r \leq d/4$

$$\left(\frac{1}{|B_r|}\int_{B_r(x_0)} u^2\right)\left(\frac{1}{|B_r|}\int_{B_r(x_0)} u^{-\frac{2}{p-1}}\right)^{p-1} \leq C\,.$$

Concluding remarks

Quantitative estimates of unique continuation seem to be a necessary ingredient for stability estimates for IP with interior data, when available data depend on few solutions of the direct problem, and few restrictions can be imposed on such solutions.

Open issues may arise in investigating the vanishing rate of gradients or Jacobians.

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Concluding remarks

For equations with no zero order term, such as

$$div(\sigma \nabla u) = 0 \ ,$$

with $\lambda^{-1} I \leq \sigma \leq \lambda I$, and if $n \geq 3$, $\sigma \in C^{0,1}$, estimates on the vanishing rate of $|\nabla u|^2$ are available.

If also a zero order term is present, e.g.:

$$div(\sigma \nabla u) + qu = 0 \ ,$$

with $|q| \leq K$, estimates on the vanishing rate are known for $|\nabla u|^2 + |u|^2$.

The situation is not clear for $|\nabla u|^2$ alone.

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with $\lambda^{-1} I \leq \sigma \leq \lambda I$, and if $n \geq 3$, $\sigma \in C^{0,1}$, estimates on the vanishing rate of $|\nabla u|^2$ are available.
If also a zero order term is present, e.g.:

$$div(\sigma \nabla u) + qu = 0 \ ,$$

with $|q| \leq K$, estimates on the vanishing rate are known for $|\nabla u|^2 + |u|^2$.

The situation is not clear for $|\nabla u|^2$ alone.

Concluding remarks

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Concluding remarks

An example

$$u_{xx} + qu = 0 \quad \text{on } (0, +\infty) \; ,$$

with

$$q(x) = \chi_{(0, \frac{\pi}{2})}(x) \; ,$$

and

$$u(x) = \begin{cases} \sin x & \text{if } 0 < x \leq \frac{\pi}{2} \; , \\ 1 & \text{if } x \geq \frac{\pi}{2} \; . \end{cases}$$

Concluding remarks

Further problems arise with Jacobians.

Example (Laugesen '96). $\forall \varepsilon > 0\ \exists \Phi : \partial B \mapsto \partial B$ homeomorphism, such that $|\Phi(x) - x| < \varepsilon, \forall x \in \partial B$ and the solution $U = (u_1, u_2, u_3)$ to

$$\begin{cases} \Delta U = 0, & \text{in} \quad B, \\ U = \Phi, & \text{on} \quad \partial B. \end{cases}$$

is not one-to-one.

Concluding remarks

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Concluding remarks

Example (Jin and Kazdan '91).
$\exists \sigma \in C^\infty(\mathbb{R}^3)$, $\lambda^{-1} I \leq \sigma \leq \lambda I$ and a solution $U = (u_1, u_2, u_3)$ to

$$div(\sigma \nabla U) = 0 \text{ in } \mathbb{R}^3 ,$$

such that

$$\begin{cases} detDU = 0, & \text{for} \quad x_3 \leq 0 , \\ detDU > 0, & \text{for} \quad x_3 > 0 . \end{cases}$$

The end.

THANKS!