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Invertible Harmonic Mappings in the Plane

Giovanni Alessandrini¹ Vincenzo Nesi²

¹ M Università di Trieste

²Università La Sapienza di Roma

The 4th Symposium on Analysis & PDEs, Purdue 2009

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The Basic Question

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Let $B \subset \mathbb{R}^2$ be the unit disk. Let $D \subset \mathbb{R}^2$ be a Jordan domain.

Given a homeomorphism $\Phi : \partial B \mapsto \partial D$, consider the solution $U = (u_1, u_2) : B \mapsto \mathbb{R}^2$ to the following Dirichlet problem

$$\left\{ egin{array}{ccc} \Delta U=0, & {
m in} & B, \ U=\Phi, & {
m on} & \partial B. \end{array}
ight.$$

Under which conditions on Φ do we have that U is a homeomorphism of $\overline{B} \mapsto \overline{D}$?

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The Classical Results

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 $\Phi: \partial B \mapsto \partial D,$ $\begin{cases} \Delta U = 0, \text{ in } B, \\ U = \Phi, \text{ on } \partial B. \end{cases}$

Theorem (H. Kneser '26)

If D is convex, then U is a homeomorphism of \overline{B} onto \overline{D} .

Posed as a problem by Radó ('26), rediscovered by Choquet ('45).

Theorem (H. Lewy '36) If $U : B \mapsto \mathbb{R}^2$ is a harmonic homeomorphism, then it is a diffeomorphism.

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The Classical Results

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• What happens in higher dimensions?

• Can we replace Δ with other elliptic operators?

- Can we replace the diagonal △ system with other elliptic systems?
- Can we dispense with the convexity of the target D?

Natural questions.

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- Minimal surfaces.
- Inverse problems.
- Homogenization.
- Variational grid generation.

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Higher Dimensions.

- Wood ('74): There exists a harmonic homeomorphism
 U : ℝ³ → ℝ³ such that det DU(0) = 0.
- Melas ('93): There exists a harmonic homeomorphism $U: \overline{B} \mapsto \overline{B}, B \subset \mathbb{R}^3$ unit ball, such that det DU(0) = 0.
- Laugesen ('96): ∀ε > 0 ∃Φ : ∂B ↦ ∂B homeomorphism, such that |Φ(x) x| < ε, ∀x ∈ ∂B and the solution U to

$$\begin{cases} \Delta U = 0, \text{ in } B, \\ U = \Phi, \text{ on } \partial B. \end{cases}$$

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Bauman-Marini-Nesi ('01):

1

$$\operatorname{div}(\sigma \nabla u_i) = \mathbf{0}, i = \mathbf{1}, \mathbf{2},$$

Elliptic Operators.

$$\sigma = \{\sigma_{ij}\} \;,\; \textit{K}^{-1}\textit{I} \leq \sigma \leq \textit{K}\textit{I} \;,\; \sigma \in \textit{C}^{lpha} \;.$$

the Kneser and the Lewy theorems continue to hold. • A.-Nesi ('01):

$$\sigma = \{\sigma_{ij}\}, \ K^{-1}I \le \sigma \le KI, \ \sigma \in L^{\infty}$$

the Kneser theorem holds true the Lewy theorem is replaced with

Theorem If $U: B \mapsto \mathbb{R}^2$ is a σ -harmonic homeomorphism, then

 $\log |\det DU| \in BMO$.

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Elliptic Operators.

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A.-Sigalotti('01):

$$div(|\sigma \nabla u_i \cdot \nabla u_i|^{\frac{p-2}{2}} \sigma \nabla u_i) = 0, i = 1, 2, p > 1,$$

$$\sigma = \{\sigma_{ij}\}, \ K^{-1}I \le \sigma \le KI, \ \sigma \in C^{0,1}.$$

the Kneser and the Lewy theorems continue to hold.

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Elliptic Systems.

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• Harmonic mappings between Riemann surfaces, Shoen and Yau ('78), Jost ('81).

 $\begin{cases} \operatorname{div}(M \nabla u_1 + N \nabla u_2) = 0, \\ \operatorname{div}(P \nabla u_1 + Q \nabla u_2) = 0. \end{cases}$

M, N, P, Q are 2 × 2 real constant symmetric matrices. Legendre–Hadamard condition

 $\eta_1^2 M \xi \cdot \xi + \eta_1 \eta_2 (N+P) \xi \cdot \xi + \eta_2^2 Q \xi \cdot \xi > 0, \quad \forall \xi, \eta \in \mathbb{R}^2 \setminus \{0\}.$

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Elliptic Systems. Equivalence.

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We say

$$\begin{cases} \operatorname{div}(M \nabla u_1 + N \nabla u_2) = 0, \\ \operatorname{div}(P \nabla u_1 + Q \nabla u_2) = 0, \end{cases} \sim \begin{cases} \operatorname{div}(M' \nabla u_1 + N' \nabla u_2) = 0, \\ \operatorname{div}(P' \nabla u_1 + Q' \nabla u_2) = 0, \end{cases}$$

f there exists a non-singular 2 imes 2 matrix $\left(egin{array}{cc} lpha & eta \\ \gamma & \delta \end{array}
ight)$ such

 $\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{pmatrix} \alpha \mathrm{Id} & \beta \mathrm{Id} \\ \gamma \mathrm{Id} & \delta \mathrm{Id} \end{pmatrix} \begin{pmatrix} M' & N' \\ P' & Q' \end{pmatrix}$

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Elliptic Systems. Equivalence.

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$$\begin{cases} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0, \end{cases} \sim \begin{cases} \operatorname{div}(M'\nabla u_1 + N'\nabla u_2) = 0, \\ \operatorname{div}(P'\nabla u_1 + Q'\nabla u_2) = 0, \end{cases}$$

if there exists a non-singular 2 × 2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that

 $\left(\begin{array}{cc} M & N \\ P & Q \end{array}\right) = \left(\begin{array}{cc} \alpha \mathrm{Id} & \beta \mathrm{Id} \\ \gamma \mathrm{Id} & \delta \mathrm{Id} \end{array}\right) \left(\begin{array}{cc} M' & N' \\ P' & Q' \end{array}\right).$

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Elliptic Systems.

The Kneser theorem fails.

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A.-Nesi ('09). Either

$$\begin{array}{l} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0, \end{array} \sim \begin{cases} \operatorname{div}(M\nabla u_1) = 0, \\ \operatorname{div}(M\nabla u_2) = 0, \end{cases} \text{ (pure diag)} \end{array}$$

or

there exists a polynomial solution U to

 $\begin{aligned} \operatorname{div}(M\nabla u_1 + N\nabla u_2) &= 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) &= 0, \end{aligned}$

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Elliptic Systems. Examples.

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For any ε > 0 the system

$$\begin{cases} u_{1,xx} + u_{1,yy} = 0, \\ (1 + \varepsilon)u_{2,xx} + u_{2,yy} = 0, \end{cases}$$

is not equivalent to a pure diagonal system.

• The Lamé system

 $\mu \operatorname{div}((DU)^T + DU) + \lambda \nabla(\operatorname{div} U) = 0.$

 $\mu, \lambda \in \mathbb{R}$ with $\mu > 0$ and $\mu + \lambda > 0$ is not equivalent to a pure diagonal system.

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The Lamé system

$$\mu \operatorname{div}((DU)^{T} + DU) + \lambda \nabla(\operatorname{div} U) = 0.$$

 $\mu, \lambda \in \mathbb{R}$ with $\mu > 0$ and $\mu + \lambda > 0$ is not equivalent to a pure diagonal system.

Elliptic Systems.

The Kneser theorem fails for Lamé, $\mu = \lambda = 1$

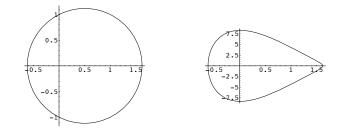


Figure: ∂B and its image $\Phi(\partial B)$.

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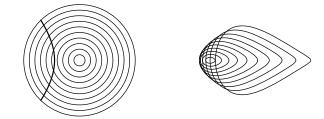


Figure: Left: circles C_r of varying radii and the nodal line of the Jacobian (an hyperbola) drawn within *B*. Right: the images $U(C_r)$.

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$\Phi: \partial B \mapsto \partial D,$ $\Delta U = 0, \quad \text{in} \quad B.$

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- $\left\{ \begin{array}{ll} \Delta U=0, & \text{in} \quad B, \\ U=\Phi, & \text{on} \quad \partial B. \end{array} \right.$
- Choquet ('45): If *D* is not convex, then there exists a homeomorphism Φ : ∂B → ∂D such that *U* is not one-to-one.
- A:-Nesi ('09): another example.

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Non-convex Target.

Given D, possibly non-convex,

• what are the additional conditions on the homeomorphism

$$\Phi:\partial B\mapsto \partial D,$$

such that the solution U to

$$\left\{ \begin{array}{ll} \Delta U = 0, & \text{in} \quad B, \\ U = \Phi, & \text{on} \quad \partial B. \end{array} \right.$$

is a homeomorphism of $\overline{B} \mapsto \overline{D}$?

• assume in addition $U \in C^1(\overline{B}; \mathbb{R}^2)$, under which conditions on Φ do we have that U is a diffeomorphism of $\overline{B} \mapsto \overline{D}$?

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 assume in addition U ∈ C¹(B; R²), under which conditions on Φ do we hav

U is a diffeomorphism of $\overline{B} \mapsto \overline{D}$?

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The necessary condition.

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If U is an orientation preserving diffeomorphism then, in particular,

det DU > 0 everywhere on ∂B . (1) = (φ, ψ) , and denote $dg(\theta) = \frac{1}{2\pi} \text{P.V.} \int_{0}^{2\pi} \frac{g(\tau)}{\tan\left(\frac{\theta-\tau}{2}\right)} d\tau, \quad \theta \in [0, 2\pi],$

(1) is equivalent to

 $\frac{\partial \phi}{\partial \theta} H\left(\frac{\partial \psi}{\partial \theta}\right) - \frac{\partial \psi}{\partial \theta} H\left(\frac{\partial \phi}{\partial \theta}\right) > 0 \quad \text{everywhere on } \partial B. \quad (2)$

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The necessary condition.

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det
$$DU > 0$$
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Set $\Phi = (\varphi, \psi)$, and denote

$$Hg(heta) = rac{1}{2\pi} ext{ P.V.} \int_{0}^{2\pi} rac{g(au)}{ an\left(rac{ heta- au}{2}
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The main theorem.

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Theorem (A.- Nesi '09)

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1 . Let U be the solution to

$$\left\{ egin{array}{ccc} \Delta U=0, & \mbox{in} & B, \ U=\Phi, & \mbox{on} & \partial B. \end{array}
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and assume, in addition, that $U \in C^1(\overline{B}; \mathbb{R}^2)$. The mapping U is a diffeomorphism of \overline{B} onto \overline{D} if and only if

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The main theorem, remark. Let co(D) be the convex hull of D. We define the convex part of ∂D as the closed set

 $\gamma_{c} = \partial D \cap \partial (co(D)).$

We define the non–convex part of ∂D as the open set

 $\gamma_{nc} = \partial D \setminus \partial(co(D)).$

emma

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1 , and assume that $U \in C^1(\overline{B}; \mathbb{R}^2)$. We always have

det DU > 0 everywhere on $\Phi^{-1}(\gamma_c)$.

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Proof: Hopf lemma

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Proof: Hopf lemma

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The main theorem, improved.

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Theorem (A.- Nesi '09)

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1 . Let U be the solution to

$$\left\{ \begin{array}{ll} \Delta U = 0, & \text{in} \quad B, \\ U = \Phi, & \text{on} \quad \partial B. \end{array} \right.$$

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The main theorem, improved.

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The main theorem, proof (i).

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We assume

det DU > 0 everywhere on ∂B .

The crucial point is to prove that

det DU > 0 everywhere in B.

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The main theorem, proof (i).

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The Jacobian may change sign.

a polynomial example (i)

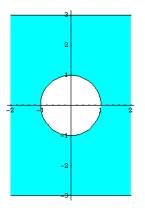


Figure:
$$u_1 = \Re\{\frac{(z+1)^2 - 1}{2}\}, u_2 = \Im\{\frac{1 - (z-1)^2}{2}\}$$

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The Jacobian may change sign.

a polynomial example (ii)

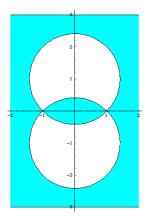


Figure: $u_1 = \Re\{(z+1)^3\}, u_2 = \Im\{(z-1)^3\}$

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The main theorem, proof (ii).

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The condition

 $\det DU > 0 \text{ everywhere in } B,$

is equivalent to

 $abla(au_1 + bu_2) \neq 0$ everywhere in B.

for every $(a, b) \neq (0, 0)$.

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The main theorem, proof (iii).

• Fix (*a*, *b*) and denote $u = au_1 + bu_2$, \tilde{u} its harmonic conjugate and

$$f = u + i\tilde{u}$$

Denote

$$WN(f(\partial B)) = \frac{1}{2\pi} \int_{\partial B} d \arg\left(\frac{\partial f}{\partial \theta}\right).$$

The argument principle says
 WN(f(∂B)) = ♯ critical points of u + 1.

• We prove

 $\operatorname{WN}(f(\partial B)) = \operatorname{WN}(\Phi(\partial B)) = 1.$

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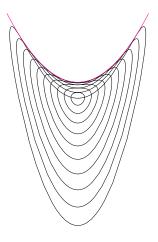
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The Counterexample. $U(x, y) = (x, x^2 - y^2)$.



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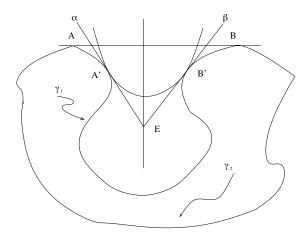
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The Counterexample, continued.



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• What if $\Phi : \partial B \mapsto \partial D$ is only a homeomorphism?

- Can we replace Δ with div($\sigma \nabla \cdot$)?
- Higher dimensions?

Open issues.

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Thanks!

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